

Improvement of a provisional solution of the quantum-corrected field equations of $d = 11$ supergravity on flat \mathbf{R}^4 times a compact hyperbolic 7-manifold, and modes that decay along the beam line outside the interaction region at the LHC

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Abstract

A recent provisional solution of the quantum-corrected field equations of $d = 11$ supergravity on flat \mathbf{R}^4 times a compact hyperbolic 7-manifold \bar{H}^7 , in the presence of magnetic 4-form fluxes wrapping 4-cycles of \bar{H}^7 , is improved by showing that the curvature radius B of \bar{H}^7 and the r.m.s. 4-form flux strength h each have a single stationary point as the field redefinition parameter c is varied. Application of the principle of minimal sensitivity then fixes c in a moderate range such that B and $h^{1/3}$ vary by only 4% and 5% respectively over this range. The new best value of B is $\simeq 0.28\kappa_{11}^{2/9} \simeq 1.2M_{11}^{-1}$. The low-lying bosonic Kaluza-Klein modes of the bulk are studied. The classically massless harmonic 3-form modes of the 3-form gauge field, whose number is estimated as roughly 10^{32} if the intrinsic volume of \bar{H}^7 is $\sim 10^{35}$, acquire a mass $\simeq 0.2\frac{A}{B}$ from quantum corrections, where the warp factor A is fixed by the boundary conditions at the Hořava-Witten (HW) boundary to lie between about 0.7 and 0.9. They have axion-like couplings to the SM gauge bosons. Their lifetimes range from about 10^{-27} seconds to several hours depending on the distance of their centre from the HW boundary, and they can decay along the beam line outside the interaction region at the LHC, with a distribution that shows a power law rather than exponential decrease with distance from the IP. Approximate exclusion limits are obtained from recent LHC data, and discovery prospects at ATLAS and CMS are studied.

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1 Introduction

The weakness of the gravitational interaction relative to the strong and electroweak interactions would have a natural explanation if there existed $n \geq 2$ compact extra spatial dimensions of volume $\sim 10^{31} \text{ TeV}^{-n}$ in which the gravitational field is diluted, while the Standard Model (SM) fields are confined to a very small part of this region. The mass defining the strength of gravity in $4 + n$ dimensions would then be around a TeV [1, 2, 3].

The confinement of the SM fields to a small part of the $3 + n$ spatial dimensions is natural in $d = 11$ supergravity [4], because the $d = 11$ supergravity multiplet does not couple to any matter fields in $10 + 1$ smooth dimensions, but can couple to matter fields on various types of localized impurity, the simplest of which is a Hořava-Witten (HW) boundary [5, 6, 7, 8, 9, 10, 11].

If the 7 compact dimensions have the topology of a compact orientable hyperbolic 7-manifold \bar{H}^7 that admits a spin structure [12], then when the metric on them is locally maximally symmetric, and their curvature is fixed, they are completely rigid [13, 14, 15, 16, 17]. Their shape and size is determined by their topology, and they

can be arbitrarily large. The number of distinct topologies for which their volume is $< V$ grows as $V^{\sigma V}$ at large V , where $\sigma > 0$ is a constant [18, 19]. The SM fields can be accommodated on a HW boundary $\mathbf{R}^4 \times \bar{H}^6$ of $\mathbf{R}^4 \times \bar{H}^7$, with a closed hyperbolic Cartesian factor \bar{H}^6 .

I shall use the notation and results of [20], with some improvements as follows.

In section 3 of [20], a provisional solution of the quantum-corrected Einstein equations of $d = 11$ supergravity on flat \mathbf{R}^4 times \bar{H}^7 , in the presence of magnetic 4-form fluxes $H_{IJKL} = \partial_I C_{JKL} - \partial_J C_{KLI} + \partial_K C_{LIJ} - \partial_L C_{IJK}$ of the 3-form gauge field C_{IJK} of $d = 11$ supergravity wrapping 4-cycles of \bar{H}^7 , was obtained in the approximation of working to leading order in the Lukas-Ovrut-Waldram (LOW) harmonic expansion of the energy-momentum tensor on \bar{H}^7 [21]. The 4-form fluxes were assumed to be proportional to harmonic 4-forms on \bar{H}^7 , and thus to solve the classical CJS field equations for H_{IJKL} , and the quantum corrections to those field equations were neglected. The fluxes were assumed to be approximately uniformly distributed across \bar{H}^7 , so that the LOW expansion only needed to be applied over relatively small local regions of \bar{H}^7 , and to leading order in the LOW expansion, the flux bilinears were assumed to have the form:

$$H_{ABEF}H_{CDGH}G^{EG}G^{FH} = \frac{h^2}{B^8}(G_{AC}G_{BD} - G_{BC}G_{AD}), \quad (1)$$

where $h \geq 0$ is a constant of dimension length³, B is the curvature radius of \bar{H}^7 , and G_{IJ} is the $d = 11$ metric, which on \bar{H}^7 has the form $G_{AB} = B^2\bar{g}_{AB}$, where \bar{g}_{AB} is a metric of constant sectional curvature -1 on \bar{H}^7 . Coordinate indices I, J, K, \dots run over all 11 dimensions, coordinate indices μ, ν, σ, \dots are tangential to the four extended space-time dimensions, and coordinate indices A, B, C, \dots are tangential to \bar{H}^7 . The coordinates are $x^I = (\tilde{x}^\mu, \bar{x}^A)$. The metric is mostly $+$, and units such that $\hbar = c = 1$ are used. The dependence on B is fixed by the fact that H_{ABCD} is independent of B , because the integral of $H_{ABCD}d\bar{x}^A d\bar{x}^B d\bar{x}^C d\bar{x}^D$ over a 4-cycle of \bar{H}^7 is quantized, and independent of B [22, 23].

The bosonic part of the quantum-corrected action on the 11-dimensional bulk was assumed to have the form:

$$\Gamma_{\text{SG}}^{(\text{bos})} = S_{\text{CJS}}^{(\text{bos})} + \Gamma_{\text{SG}}^{(8,\text{bos})}, \quad (2)$$

where $S_{\text{CJS}}^{(\text{bos})}$ is the bosonic part of the classical action of $d = 11$ supergravity [4]:

$$S_{\text{CJS}}^{(\text{bos})} = \frac{1}{2\kappa_{11}^2} \int_{\mathcal{B}} d^{11}x e \left(R - \frac{1}{48} H_{IJKL} H^{IJKL} - \frac{1}{144^2} \epsilon_{(11)}^{I_1 \dots I_{11}} C_{I_1 I_2 I_3} H_{I_4 \dots I_7} H_{I_8 \dots I_{11}} \right), \quad (3)$$

and for covariantly constant fluxes, the leading quantum correction $\Gamma_{\text{SG}}^{(8,\text{bos})}$ is a dimension 8 local term of the form [24]:

$$\Gamma_{\text{SG}}^{(8,\text{bos})} = \frac{1}{147456\pi^2\kappa_{11}^2} \left(\frac{\kappa_{11}}{4\pi} \right)^{4/3} \int_{\mathcal{B}} d^{11}x e \left(t_8 t_8 \check{R}^4 + Z - \frac{1}{6} \epsilon_{11} t_8 C R^4 + c (\tilde{Z} - Z) \right). \quad (4)$$

Here \mathcal{B} means the bulk, κ_{11} is the gravitational coupling constant in 11 dimensions, $e = \sqrt{-G}$ is the determinant of the vielbein $e_{I\hat{J}}$, where G is the determinant of the metric G_{IJ} , and the antisymmetric tensor $\epsilon_{(11)}^{I_1 \dots I_{11}}$ is related to the $\text{SO}(10,1)$ invariant tensor $\epsilon_{(11)}^{\hat{I}_1 \dots \hat{I}_{11}}$, with components $0, \pm 1$, by $\epsilon_{(11)}^{I_1 \dots I_{11}} = e^{I_1 \hat{J}_1} \dots e^{I_{11} \hat{J}_{11}} \epsilon_{(11)}^{\hat{J}_1 \dots \hat{J}_{11}}$. Hatted indices are local Lorentz indices. The Riemann tensor for the metric G_{IJ} is defined by:

$$R_{IJ}{}^K{}_L = \partial_I \Gamma_J{}^K{}_L - \partial_J \Gamma_I{}^K{}_L + \Gamma_I{}^K{}_M \Gamma_J{}^M{}_L - \Gamma_J{}^K{}_M \Gamma_I{}^M{}_L, \quad (5)$$

where the standard Christoffel connection is $\Gamma_I{}^J{}_K = \frac{1}{2} G^{JL} (\partial_I G_{LK} + \partial_K G_{LI} - \partial_L G_{IK})$, and the Ricci tensor and scalar are defined by $R_{IJ} = R_{KI}{}^K{}_J$, and $R = G^{IJ} R_{IJ}$. The modified Riemann tensor \check{R}_{IJKL} is defined by:

$$\check{R}_{IJKL} = R_{IJKL} - \frac{1}{8} H_{IKMN} H_{JL}{}^{MN} + \frac{1}{8} H_{JKMN} H_{IL}{}^{MN}. \quad (6)$$

The notation $t_8 t_8 \check{R}^4$ is a shorthand for

$$t_8^{I_1 I_2 J_1 J_2 K_1 K_2 L_1 L_2} t_8^{M_1 M_2 N_1 N_2 O_1 O_2 P_1 P_2} \check{R}_{I_1 I_2 M_1 M_2} \check{R}_{J_1 J_2 N_1 N_2} \check{R}_{K_1 K_2 O_1 O_2} \check{R}_{L_1 L_2 P_1 P_2}, \quad (7)$$

where $t_8^{IJKLMNOP}$ is a tensor built from G^{IJ} and antisymmetric in each successive pair of indices, such that for antisymmetric tensors A_{IJ} , B_{IJ} , C_{IJ} , and D_{IJ} :

$$\begin{aligned} t_8^{IJKLMNOP} A_{IJ} B_{KL} C_{MN} D_{OP} &= \\ &= 8 (\text{tr}(ABCD) + \text{tr}(ACBD) + \text{tr}(ACDB)) \\ &\quad - 2 (\text{tr}(AB) \text{tr}(CD) + \text{tr}(AC) \text{tr}(BD) + \text{tr}(AD) \text{tr}(BC)) = \\ &= 8 (A_{IJ} B_{JK} C_{KL} D_{LI} + A_{IJ} C_{JK} B_{KL} D_{LI} + A_{IJ} C_{JK} D_{KL} B_{LI}) \\ &\quad - 2 (A_{IJ} B_{JI} C_{KL} D_{LK} + A_{IJ} C_{JI} B_{KL} D_{LK} + A_{IJ} D_{JI} B_{KL} C_{LK}) \end{aligned} \quad (8)$$

Repeated lower coordinate indices are understood to be contracted with an inverse metric, for example $A_{IJ} B_{JK} \equiv A_I{}^J B_{JK} = G^{JL} A_{IL} B_{JK}$. Thus:

$$\begin{aligned} t_8 t_8 \check{R}^4 &= 12 \check{R}_{IJMN} \check{R}_{IJMN} \check{R}_{KLOP} \check{R}_{KLOP} + 24 \check{R}_{IJMN} \check{R}_{IJOP} \check{R}_{KLOP} \check{R}_{KLMN} \\ &\quad - 96 \check{R}_{IJMN} \check{R}_{IJMP} \check{R}_{KLON} \check{R}_{KLOP} - 48 \check{R}_{IJMN} \check{R}_{IJOP} \check{R}_{KLMP} \check{R}_{KLON} \\ &\quad - 96 \check{R}_{IJMN} \check{R}_{KJMN} \check{R}_{ILOP} \check{R}_{KLOP} - 48 \check{R}_{IJMN} \check{R}_{KLMN} \check{R}_{ILOP} \check{R}_{KJOP} \\ &\quad + 192 \check{R}_{IJMN} \check{R}_{KJON} \check{R}_{KLOP} \check{R}_{ILMP} + 384 \check{R}_{IJMN} \check{R}_{ILMP} \check{R}_{KJOP} \check{R}_{KLON}. \end{aligned} \quad (9)$$

Similarly:

$$\begin{aligned} \frac{1}{6}\epsilon_{11}t_8CR^4 \equiv & 4\epsilon_{(11)}^{I_1\dots I_{11}}C_{I_1I_2I_3}R_{I_4I_5JK}R_{I_6I_7KL}R_{I_8I_9LM}R_{I_{10}I_{11}MJ} \\ & - \epsilon_{(11)}^{I_1\dots I_{11}}C_{I_1I_2I_3}R_{I_4I_5JK}R_{I_6I_7KJ}R_{I_8I_9LM}R_{I_{10}I_{11}ML}. \end{aligned} \quad (10)$$

And:

$$\begin{aligned} Z \equiv & -\frac{1}{4!}\epsilon_{11}\epsilon_{11}\check{R}^4 \equiv -\frac{1}{4!}\epsilon_{(11)}^{IJKL_1L_2M_1M_2N_1N_2O_1O_2}\epsilon_{(11)IJKP_1P_2Q_1Q_2R_1R_2S_1S_2} \\ & \times \check{R}_{L_1L_2}^{P_1P_2}\check{R}_{M_1M_2}^{Q_1Q_2}\check{R}_{N_1N_2}^{R_1R_2}\check{R}_{O_1O_2}^{S_1S_2} \\ & = \frac{8!}{4}\check{R}_{L_1L_2}^{[L_1L_2}\check{R}_{M_1M_2}^{M_1M_2}\check{R}_{N_1N_2}^{N_1N_2}\check{R}_{O_1O_2}^{O_1O_2]}, \end{aligned} \quad (11)$$

\tilde{Z} is the result of using the classical Einstein equations following from (3):

$$R_{IJ} - \frac{1}{2}RG_{IJ} - \frac{1}{12}H_I^{KLM}H_{JKLM} + \frac{1}{96}H^{KLMN}H_{KLMN}G_{IJ} = 0, \quad (12)$$

to replace all Ricci tensors and Ricci scalars resulting from writing out the antisymmetrization of the upper indices, in the final form of (11), by bilinears in H_{IJKL} , and c is a coefficient.

The coefficient of the $\epsilon_{11}t_8CR^4$ term in (4), which is known as the Green-Schwarz term because of its role in anomaly cancellation [25], is fixed absolutely by anomaly cancellation on five-branes [26, 27, 28, 29, 30], and confirmed by anomaly cancellation in Hořava-Witten theory [31, 32, 33, 34, 35, 36, 29, 37, 10].

The relative coefficients of all terms in (4) are fixed by supersymmetry, up to the fact that arbitrary multiples of linear combinations of terms that vanish when the classical Einstein equations (12), and the classical equations:

$$D_L H^{LIJK} - \frac{1}{3456}\epsilon_{(11)}^{IJKLMNQPQRS}H_{LMNO}H_{PQRS} = 0 \quad (13)$$

for C_{IJK} that follow from (3), are satisfied can be added, because the overall coefficients of such linear combinations of terms can be adjusted arbitrarily by making redefinitions of the fields of the form $G_{IJ}, C_{IJK} \rightarrow G_{IJ} + \kappa_{11}^{4/3}X_{IJ}, C_{IJK} + \kappa_{11}^{4/3}Y_{IJK}$, where X_{IJ} and Y_{IJK} are dimension 6 polynomials in the fields and their derivatives [38, 39, 40]. The $c(\tilde{Z} - Z)$ term has been included to allow for this ambiguity. When $c = 0$, compactification of (4) on a small S^1 gives the form that arises naturally from type IIA superstring scattering amplitudes [41, 42], while $c = 1$ gives the form of (4) that arises

naturally when supersymmetry is systematically implemented by the Noether method [43, 38, 39, 44].

Field redefinitions of this type are like a change of coordinates in “field space”, so they do not change the physical content of the theory. In particular, they do not alter the S -matrix [45, 46, 47]. The collection of all such field redefinitions, with X_{IJ} and Y_{IJK} generalized to expansions of the form $\sum_{n \geq 0} \kappa_{11}^{2n/3} X_{IJ}^{(n)}$ and $\sum_{n \geq 0} \kappa_{11}^{2n/3} Y_{IJK}^{(n)}$, where $X_{IJ}^{(n)}$ and $Y_{IJK}^{(n)}$ are dimension $6 + 3n$ polynomials in the fields and their derivatives, forms a “field redefinition group”, that generalizes the renormalization group of renormalizable quantum field theories.

At low orders of perturbation theory field redefinitions do affect physical quantities, and can even affect whether a particular type of solution of the quantum-corrected field equations exists or not. Sensitivity to field redefinitions should decrease as higher-order corrections are included, so at low orders of perturbation theory, we should use the *principle of minimal sensitivity* (PMS) [48] to choose the best field redefinition, as in perturbative QCD. Doing that should minimize the size of the higher-order corrections. In QCD the PMS resolves the renormalization scheme ambiguity, and here it means using “coordinates in field space” best suited to the geometry being studied.

With the exception of the $\epsilon_{11} t_8 C R^4$ term, and the 4-field parts of terms that depend on H_{IJKL} through $D_I H_{JKLM}$, and thus vanish for covariantly constant fluxes [49, 50, 51, 40], the C_{IJK} -dependent terms in $\Gamma_{\text{SG}}^{(8, \text{bos})}$ are not yet known. Their inclusion through the modified Riemann tensor \check{R}_{IJKL} defined in (6) is a guess such that if (4) is compactified on a small S^1 , such that the fields are covariantly constant and only the fields of the type I supergravity multiplet in 10 dimensions are nonzero, it agrees with the Kehagias-Partouche (KP) conjecture for the completion of the dimension 8 local term in 10 dimensions [52, 53], up to correction terms that contain factors that occur in the classical Einstein equation in 10 dimensions. The KP conjecture is supported by recent calculations by Richards [42], but was shown in section 2 of [20] to require a correction, because it cannot be oxidized as it stands to a generally covariant formula in 11 dimensions.

The metric in the bulk, away from the immediate vicinity of the HW boundary, is assumed to have the form:

$$ds_{11}^2 = G_{IJ} dx^I dx^J = A^2 \eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu + B^2 \bar{g}_{AB} d\bar{x}^A d\bar{x}^B, \quad (14)$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the metric on $(3 + 1)$ -dimensional Minkowski space,

and A is a constant.

Because the leading quantum correction (4) is a local term, independent of the topology of \bar{H}^7 , the assumption (1) means that the quantum-corrected Einstein equations at this order are consistent with the locally maximally symmetric metric ansatz (14), so by Palais's Principle of Symmetric Criticality [54, 55, 56], the Einstein equations can be derived by substituting (14) into the action (2), and varying with respect to the constants A and B .

More directly, when we expand the quantum-corrected action (2) for a general perturbation $\bar{\bar{G}}_{IJ} \equiv G_{IJ} + 2h_{IJ}$ of the metric ansatz (14) in powers of the perturbation tensor h_{IJ} , the result is a sum of local terms built from G_{IJ} , its Riemann tensor R_{IJKL} , the covariant derivatives D_I built from G_{IJ} that satisfy $D_I G_{JK} \equiv 0$, and the tensor h_{IJ} , and for the terms linear in h_{IJ} we can remove all covariant derivatives from h_{IJ} by integrations by parts, and those covariant derivatives then give 0 by the local symmetry of (14). The quantum-corrected Einstein equations for (14) are thus proportional to the metric blocks $G_{\mu\nu}$ and G_{AB} corresponding to the irreducible locally symmetric space Cartesian factors of the 11-dimensional product space, and the proportionality factors can be obtained by using, for example, $\frac{\delta\Gamma_{\text{SG}}^{(\text{bos})}}{\delta B} = \frac{\delta G_{AB}}{\delta B} \frac{\delta\Gamma_{\text{SG}}^{(\text{bos})}}{\delta G_{AB}} = \frac{2}{B} G_{AB} \frac{\delta\Gamma_{\text{SG}}^{(\text{bos})}}{\delta G_{AB}}$.

In terms of rescaled parameters:

$$\tilde{B} \equiv \frac{2^{\frac{29}{18}} \pi^{\frac{5}{9}}}{21^{\frac{1}{3}} \kappa_{11}^{2/9}} \frac{B}{\kappa_{11}^{2/9}}, \quad \tilde{h} \equiv \frac{2^{\frac{29}{6}} \pi^{\frac{5}{3}}}{21} \frac{h}{\kappa_{11}^{2/3}}, \quad (15)$$

the action density, after substituting in the metric ansatz (14), is the square root of the determinant of \bar{g}_{AB} , times:

$$\begin{aligned} & \frac{21^{\frac{5}{3}}}{2^{\frac{163}{18}} \pi^{\frac{25}{9}} \kappa_{11}^{8/9}} A^4 \tilde{B}^7 \left(-\frac{42}{\tilde{B}^2} - \frac{7\tilde{h}^2}{8\tilde{B}^8} + \left(\frac{1}{\tilde{B}^2} + \frac{\tilde{h}^2}{8\tilde{B}^8} \right)^4 \right. \\ & \left. + c \left(-\frac{19}{21\tilde{B}^8} + \frac{89\tilde{h}^2}{189\tilde{B}^{14}} + \frac{23\tilde{h}^4}{10368\tilde{B}^{20}} + \frac{20395\tilde{h}^6}{870912\tilde{B}^{26}} + \frac{2225255\tilde{h}^8}{3009871872\tilde{B}^{32}} \right) \right). \end{aligned} \quad (16)$$

This is obtained in the approximation of using (1) for the HH terms in (6) to obtain $\check{R}_{ABCD} = \left(\frac{1}{\tilde{B}^2} + \frac{\tilde{h}^2}{8\tilde{B}^8} \right) (G_{AD}G_{BC} - G_{AC}G_{BD})$, and then substituting this into (4), together with the corresponding treatment of \tilde{Z} , instead of first expanding (4) in powers of H and summing over all pairings of factors of H before using (1) and the corresponding equation with no index contractions, as would be required for H to be a Gaussian random variable with mean zero and mean square fixed by (1).

Defining:

$$\eta \equiv \frac{\tilde{h}}{\tilde{B}^3} = \frac{h}{B^3}, \quad (17)$$

the field equations reduce to:

$$\begin{aligned} & (2633637888\eta^2 + 126414618624) \tilde{B}^6 - 734832\eta^8 - 23514624\eta^6 \\ & - 282175488\eta^4 - 1504935936\eta^2 - 3009871872 - (2225255\eta^8 \\ & + 70485120\eta^6 + 6676992\eta^4 + 1417347072\eta^2 - 2723217408) c = 0, \end{aligned} \quad (18)$$

and

$$\begin{aligned} & (2633637888\eta^2 - 632073093120) \tilde{B}^6 - 18370800\eta^8 - 446777856\eta^6 \\ & - 3668281344\eta^4 - 10534551552\eta^2 - 3009871872 - (55631375\eta^8 \\ & + 1339217280\eta^6 + 86800896\eta^4 + 9921429504\eta^2 - 2723217408) c = 0. \end{aligned} \quad (19)$$

Solving these as two simultaneous linear equations for c and \tilde{B}^6 , we find:

$$c = -\frac{734832}{\mathcal{P}(\eta)} (\eta^2 + 8)^3 (\eta^4 + 60\eta^2 + 96), \quad (20)$$

$$\tilde{B}^6 = \frac{\eta^2 (\eta^2 + 8)^3 (22595\eta^6 + 52446528\eta^4 - 129537792\eta^2 + 694738944)}{448\mathcal{P}(\eta)}, \quad (21)$$

where

$$\begin{aligned} \mathcal{P}(\eta) \equiv & 2225255\eta^{10} + 186379140\eta^8 + 3386624256\eta^6 \\ & + 594708480\eta^4 + 34016329728\eta^2 - 32678608896. \end{aligned} \quad (22)$$

The polynomial $\mathcal{P}(\eta)$ is positive for $\eta^2 > \eta_{\min}^2$ and negative for $\eta^2 < \eta_{\min}^2$, where $\eta_{\min} \simeq 0.9364$, and has no real zeros other than $\eta = \pm\eta_{\min}$. c is a monotonically increasing function of η for $\eta > \eta_{\min}$, and tends to the limit $c_{\max} \equiv -\frac{734832}{2225255} \simeq -0.3302$ as $\eta \rightarrow +\infty$. The product $\tilde{B}^6 \mathcal{P}(\eta)$ is ≥ 0 for all η , so a solution with real \tilde{B} only exists for $\eta^2 > \eta_{\min}^2$, and thus only for $c < c_{\max}$. \tilde{B} and \tilde{h} are positive, so we only need to consider the region $\eta > \eta_{\min}$, and in this region, (20) determines η implicitly as a function of the field redefinition parameter c , and (21) then determines \tilde{B} as a function of c .

1.1 Application of the Principle of Minimal Sensitivity

If the system is physically sensible, then by the PMS, there should be values of the field redefinition parameter c , not too far apart, where the solution exists, and $\frac{d\tilde{B}}{dc}$ and $\frac{d\tilde{h}}{dc}$ are respectively 0. c should then be chosen somewhere between these values, in order to minimize the size of the higher order corrections. The PMS was not applied properly in [20], because only the dependence of η , there called x , on c was considered, and it was then necessary to make an ad hoc choice of c .

However $\frac{d\tilde{B}}{d\eta}$ and $\frac{d\tilde{h}}{d\eta}$, and consequently also $\frac{d\tilde{B}}{dc}$ and $\frac{d\tilde{h}}{dc}$, each have exactly one zero for $\eta > \eta_{\min}$. For $\frac{d\tilde{B}}{d\eta}$ and $\frac{d\tilde{B}}{dc}$, the zero is at $\eta \simeq 1.700$, which corresponds to $c \simeq -1.590$, $\tilde{B} \simeq 0.580$, $\tilde{h} \simeq 0.332$, $B \simeq 0.277\kappa_{11}^{2/9}$, and $h \simeq 0.0363\kappa_{11}^{2/3}$, and for $\frac{d\tilde{h}}{d\eta}$ and $\frac{d\tilde{h}}{dc}$, the zero is at $\eta \simeq 1.291$, which corresponds to $c \simeq -3.083$, $\tilde{B} \simeq 0.605$, $\tilde{h} \simeq 0.286$, $B \simeq 0.289\kappa_{11}^{2/9}$, and $h \simeq 0.0312\kappa_{11}^{2/3}$. These values give the range of values of the field redefinition parameter c , and the physical quantities B and h , selected by the PMS at this order of perturbation theory. The unphysical parameter c varies by almost a factor of 2 over this range, but B and $h^{1/3}$ vary by only 4% and 5% respectively over this range. The ad hoc value of c chosen in [20] does not lie in this range, so it is necessary to reconsider some of the conclusions of [20].

The mean of the values of c at the ends of the selected interval is $c \simeq -2.337$, which corresponds to $\eta \simeq 1.425$, $B \simeq 0.282\kappa_{11}^{2/9}$, and $h \simeq 0.0320\kappa_{11}^{2/3}$. I shall use $B \simeq 0.28\kappa_{11}^{2/9}$ as the best value of B .

The conclusion on page 42 of [20] that b_1 , the curvature radius of the closed hyperbolic Cartesian factor \bar{H}^6 of the HW boundary, lies in the range $0.97B$ to $1.00B$ is unaltered. Thus from equation (107) on that page, the Giudice-Rattazzi-Wells perturbativity criterion [57] is still satisfied by a large margin, both in the bulk and on the HW boundary.

The second derivative of (16) with respect to \tilde{B} , when (20) and (21) are satisfied, is:

$$-\frac{3^{\frac{14}{3}}7^{\frac{5}{3}}\eta^2(\eta^2+8)^2A^4}{2^{\frac{253}{18}}\pi^{\frac{25}{9}}\kappa_{11}^{8/9}\tilde{B}^3\mathcal{P}(\eta)}\left(22595\eta^{10}+36771952\eta^8+3107227616\eta^6\right. \\ \left.+1531643904\eta^4+1459150848\eta^2+44463292416\right) \quad (23)$$

Thus the solution is a minimum of the potential energy for all $\eta > \eta_{\min}$. I shall show in subsection 2.2, starting on page 23, that (23) gives a first approximation to the mass m_{dil} of the dilaton/radion, as seen on the HW boundary, of $m_{\text{dil}} \simeq 9\frac{A}{B} \simeq 30A\kappa_{11}^{-2/9} \simeq$

$7AM_{11}$, where the warp factor A will be found below to be fixed by the boundary conditions at the HW boundary to lie between about 0.7 and 0.9.

The diameter L of a compact manifold is by definition the maximum over all pairs of points of the manifold of the shortest geodesic distance between them. The *intrinsic volume* and *intrinsic diameter* of a compact hyperbolic manifold are its volume and diameter when the metric on it is locally maximally symmetric, with sectional curvature equal to -1 .

From equation (111) on page 44 of [20], with $b_1 \simeq B$, the intrinsic volume \bar{V}_6 of the closed hyperbolic Cartesian factor \bar{H}^6 of the HW boundary is now estimated to lie in the range from about 270000 to about 580000, corresponding to an Euler number $\chi(\bar{H}^6)$ in the range from about -16000 to about -35000 , where the uncertainty arises mainly from the uncertainty of the value α_U of the QCD fine structure constant $\alpha_s = \frac{g_s^2}{4\pi}$ at unification. Thus if \bar{H}^6 is reasonably isotropic, in the sense that it has an approximately spherical fundamental domain in 6-dimensional hyperbolic space H^6 , then from equation (9) on page 9 of [20], with $S_5 = \pi^3$, its intrinsic diameter \bar{L}_6 lies between about 5.7 and 6.0.

Moss's improved form of Hořava-Witten theory is used [8, 9, 10, 11]. In the region of the HW boundary, the coordinates x^I have the form (\tilde{x}^U, y) , where indices U, V, W, \dots are tangential to a family of hypersurfaces foliating the $(10 + 1)$ -dimensional manifold-with-boundary, one of these hypersurfaces coinciding with the boundary, and y takes a constant value on each of these hypersurfaces, with the value of y distinguishing the hypersurfaces. y takes the value y_1 on the boundary, and $y > y_1$ in the bulk. The symbol y is also used as the coordinate index for the y coordinate.

The boundary is equivalent to a double-sided mirror at $y = y_1$, such that all the fields on one side of the mirror are exactly copied, up to sign, on the other side of the mirror, with the fields at (\tilde{x}^U, y) mapped to the fields at $(\tilde{x}^U, 2y_1 - y)$. The Yang-Mills multiplet is adjacent to the mirror, but infinitesimally displaced from it, so that it has its own reflection infinitesimally on the other side of the mirror [34], and I shall represent this by writing the y coordinate of the Yang-Mills multiplet as $y = y_{1+}$.

The bosonic part of the Yang-Mills term in the semi-classical action on the HW boundary is:

$$S_{\text{YM}}^{(\text{bos})} = -\frac{1}{16\pi\kappa_{11}^2} \left(\frac{\kappa_{11}}{4\pi} \right)^{2/3} \int_{y=y_{1+}} d^{10}\tilde{x} \tilde{e} \left(\frac{1}{30} \text{tr} F_{UV} F^{UV} - \frac{1}{2} \bar{R}_{UV\hat{W}\hat{X}} \bar{R}^{UV\hat{W}\hat{X}} \right). \quad (24)$$

Here $F_{UV} = \partial_U A_V - \partial_V A_U + i[A_U, A_V]$ is the field strength of the E_8 Yang-Mills

gauge field $A_U = T^A A_U^A$ localized at $y = y_{1+}$, indices $\mathcal{A}, \mathcal{B}, \dots$ run over the 248 generators of E_8 , and the hermitian generators T^A in the fundamental/adjoint of E_8 satisfy $\text{tr} T^A T^B = 30\delta^{AB}$. In the $\text{SO}(16)$ basis for E_8 , the T^A are $-\frac{1}{2}i$ times the generators in Appendix 6.A of [58] or subsection 2.1 of [59], and in the $\text{SU}(9)$ basis for E_8 , the T^A are the generators in subsection 5.2 of [59]. $\tilde{e} = \sqrt{-\tilde{G}}$ is the determinant of the vielbein $\tilde{e}_{U\hat{V}}$, that satisfies $\tilde{e}_{U\hat{W}}\tilde{e}_V^{\hat{W}} = \tilde{G}_{UV}$, where \tilde{G}_{UV} is the induced metric on the boundary, which is obtained from G_{IJ} by dropping the row and column with an index y . The coefficient of the first term in (24) is fixed by anomaly cancellation [6, 31, 32, 33, 34, 35, 36, 29, 37, 10] and has the value found by Conrad [32], which is slightly different from the original value found by HW. The $\bar{R}_{UV\hat{W}\hat{X}}\bar{R}^{UV\hat{W}\hat{X}}$ term was derived by Moss [11], with:

$$\bar{R}_{UV}^{\hat{W}}{}_{\hat{X}} = \partial_U \bar{\omega}_V^{\hat{W}}{}_{\hat{X}} - \partial_V \bar{\omega}_U^{\hat{W}}{}_{\hat{X}} + \bar{\omega}_U^{\hat{W}}{}_{\hat{Y}} \bar{\omega}_V^{\hat{Y}}{}_{\hat{X}} - \bar{\omega}_V^{\hat{W}}{}_{\hat{Y}} \bar{\omega}_U^{\hat{Y}}{}_{\hat{X}}, \quad (25)$$

where

$$\bar{\omega}_{U\hat{V}\hat{W}} = \tilde{\omega}_{U\hat{V}\hat{W}} \pm \frac{1}{2} H_{\hat{y}U\hat{V}\hat{W}}, \quad (26)$$

and $\tilde{\omega}_{U\hat{V}\hat{W}} = e^X_{\hat{W}} (\tilde{\Gamma}_U^X e_{Y\hat{V}} - \partial_U e_{X\hat{V}})$ is the Levi-Civita connection for the vielbein $\tilde{e}_{U\hat{V}}$. The sign choice in (26) is correlated with the chirality conditions on the gravitino, gaugino, and supersymmetry variation parameter on the boundary.

The compact hyperbolic 7-manifold \bar{H}^7 of intrinsic volume around 10^{34} , with a closed hyperbolic boundary \bar{H}^6 of intrinsic volume in the range from about 3×10^5 to about 6×10^5 that accomodates the SM fields, and possibly also other closed hyperbolic boundaries that accomodate dark matter fields, is assumed to be obtained from a closed hyperbolic 7-manifold by cutting it along suitable 6-cycles, and keeping one connected component of the result.

The SM boundary is near a minimal-area 6-cycle of the compact hyperbolic 7-manifold that was cut to form the boundary, and the metric in the region of the boundary is a small perturbation of what it would have been if the boundary was not there. The metric in this region has the form:

$$ds_{11}^2 = G_{IJ} dx^I dx^J = a(y)^2 \eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu + b(y)^2 \hat{g}_{ab} d\hat{x}^a d\hat{x}^b + dy^2 \quad (27)$$

Here $a(y) \rightarrow A$ away from the boundary, and $a(y) = 1$ on the boundary. Indices a, b, c, \dots are tangential to \bar{H}^6 , so that \bar{x}^A in (14) is now (\hat{x}^a, y) , and \tilde{x}^U is $(\tilde{x}^\mu, \hat{x}^a)$. \hat{g}_{ab} is a metric of sectional curvature -1 on \bar{H}^6 .

If the boundary was not there, and $b(y)$ had its minimum value at $y = 0$, $b(y)$ would be $b = B \cosh\left(\frac{y}{B}\right)$. Then (27), with $a(y) = A$, would be in agreement with (14), for a particular choice of coordinates on this region of \bar{H}^7 . The effective energy-momentum tensor T_{IJ} in this region is calculated by requiring that this metric satisfies the classical Einstein equations with that T_{IJ} . We define $a(y) \equiv (1 + p(y)) A$, $b(y) \equiv (1 + q(y)) B \cosh\frac{y}{B}$, where $|p(y)|$ and $|q(y)|$ are assumed $\ll 1$, and substitute the perturbed metric into the Einstein equations with the effective T_{IJ} . Expanding to first order in p and q , $p(y)$ and $q(y)$ are found to satisfy:

$$\dot{p} = -\frac{5}{4}\dot{q} + \frac{5q}{4B \sinh\frac{y}{B} \cosh\frac{y}{B}} \quad (28)$$

$$\ddot{q} + 7\dot{q} \frac{\sinh\frac{y}{B}}{B \cosh\frac{y}{B}} - \frac{5q}{B^2 \cosh^2\frac{y}{B}} = 0, \quad (29)$$

where a dot denotes differentiation with respect to y . To find the solution of (28) and (29) such that p and q tend to 0 as $y \rightarrow \infty$, we define $\xi \equiv \tanh\frac{y}{B}$, so that $\xi \rightarrow 1$ as $y \rightarrow \infty$. The equations then become:

$$\frac{dp}{d\xi} = -\frac{5}{4} \frac{dq}{d\xi} + \frac{5q}{4\xi} \quad (30)$$

$$(1 - \xi^2) \frac{d^2 q}{d\xi^2} + 5\xi \frac{dq}{d\xi} - 5q = 0. \quad (31)$$

The solution is:

$$q(\xi) = k \left((1 - \xi)^{\frac{7}{2}} - \frac{5(1 - \xi)^{\frac{9}{2}}}{12} + \frac{35(1 - \xi)^{\frac{11}{2}}}{1056} + \frac{35(1 - \xi)^{\frac{13}{2}}}{18304} + \dots \right), \quad (32)$$

$$p(\xi) = -\frac{5}{4}q(\xi) - \frac{5}{4} \int_{\xi}^1 \frac{q(\xi')}{\xi'} d\xi', \quad (33)$$

where k is an arbitrary constant. $q(\xi)$ looks qualitatively like the base of a parabola centred at $\xi = 1$, and is $\simeq 0.6184k$ for $\xi = 0$, while $p(\xi)$ has a logarithmic singularity as $\xi \rightarrow 0_+$.

For a first estimate of the boundary conditions for the metric [60, 61, 62, 9], I neglected the flux terms in \bar{R}_{UVWX} , defined in (25), by assuming, if necessary, that $H_{\hat{g}UVW}$ is smaller than its average value, near the boundary. Then to leading order in the LOW harmonic expansion of the energy-momentum tensor

$$\tilde{T}^{(\text{bos})UV} = \frac{2}{\tilde{e}} \frac{\delta S_{\text{YM}}^{(\text{bos})}}{\delta \tilde{G}_{UV}} \quad (34)$$

on the boundary, and assuming that $\text{tr} F_{ac} F_b{}^c$ is a multiple of \tilde{G}_{ab} , the boundary conditions are:

$$\left. \frac{\dot{a}}{a} \right|_{y=y_{1+}} = -\rho \frac{\kappa_{11}^{2/3}}{b_1^4}, \quad \left. \frac{\dot{b}}{b} \right|_{y=y_{1+}} = \rho \frac{\kappa_{11}^{2/3}}{b_1^4}, \quad (35)$$

where the number ρ is:

$$\rho \equiv \frac{1}{\bar{V}_6} \int_{\bar{H}^6, y=y_{1+}} d^6 \hat{x} \sqrt{\hat{g}} \frac{1}{96\pi (4\pi)^{2/3}} \hat{g}^{ac} \hat{g}^{bd} \left(-\frac{1}{30} \text{tr} F_{ab} F_{cd} + \frac{1}{2} \bar{R}_{ab}{}^e{}_f \bar{R}_{cde}{}^f \right), \quad (36)$$

where $\bar{V}_6 \equiv \int_{\bar{H}^6} d^6 \hat{x} \sqrt{\hat{g}}$ is the intrinsic volume of the \bar{H}^6 Cartesian factor of the boundary. If there were no Yang-Mills fluxes on the boundary then ρ would be $\frac{60}{192\pi(4\pi)^{2/3}} \simeq 0.01840$.

Let $\xi_1 \equiv \tanh \frac{y_1}{B}$ denote the value of ξ at the boundary. The sum of the boundary conditions (35) gives:

$$(1 - \xi_1^2) \left(-\frac{1}{4} \frac{dq}{d\xi} \Big|_{\xi=\xi_1} + \frac{5q(\xi_1)}{4\xi_1} \right) + \xi_1 = 0, \quad (37)$$

where (30) has been used. The logarithmic singularity of p as $\xi \rightarrow 0_+$ means that we require $\xi_1 > 0$ for the assumption that $|p| \ll 1$ to be valid, so since $\frac{1}{k} \frac{dq}{d\xi} \leq 0$ and $\frac{1}{k} q \geq 0$ for $0 \leq \xi \leq 1$, (37) implies that $k < 0$. Using (37) to express k in terms of ξ_1 , we find that $q_1 \equiv q(\xi_1)$, as a function of ξ_1 , looks qualitatively like an upside-down parabola, with a peak value of 0 at $\xi_1 = 0$, and $\simeq -0.25$ at $\xi_1 \simeq 0.63$. And $\frac{b_1}{B} = \frac{(1+q_1)}{\sqrt{1-\xi_1^2}}$, as a function of ξ_1 , decreases smoothly from a peak value of 1 at $\xi_1 = 0$, to a minimum value $\simeq 0.967$ at $\xi_1 \simeq 0.56$, and then starts increasing at an increasing rate.

Using $B \simeq 0.28\kappa_{11}^{2/9}$ as the best value of B determined by the PMS, the second equation of (35) becomes:

$$(1 - \xi_1^2) \frac{dq}{d\xi} \Big|_{\xi=\xi_1} + \xi_1 \simeq \frac{729\rho(1 - 4q(\xi_1))}{\left(\sqrt{\frac{1+\xi_1}{1-\xi_1}} + \sqrt{\frac{1-\xi_1}{1+\xi_1}} \right)^4}, \quad (38)$$

so we require $\rho > 0$. Thus the number of vacuum Yang-Mills fluxes should be small enough for the R^2 term in (36) to outweigh the F^2 term.

Substituting for k from (37), we find from (38) that for $\rho = 0.01840$, $\xi_1 \simeq 0.391$, hence $k \simeq -0.768$, so $p_1 \equiv p(\xi_1) \simeq 0.165$, hence $A \simeq 0.858$, and $q_1 \simeq -0.103$, hence $b_1 \simeq 0.975B \simeq 0.27\kappa_{11}^{2/9}$. Thus working to first order in p and q has been justified.

Moss's derivation of the $R_{UVWX} R^{UVWX}$ term in (24) used an expansion scheme in which Ricci tensor and scalar terms, if present, would only show up at higher orders

[11]. If the $R_{UVWX}R^{UVWX}$ term was in fact the first term in a Lovelock-Gauss-Bonnet term of the form $R_{UVWX}R^{UVWX} - 4R_{UV}R^{UV} + R^2$, the size of the RR terms in (36) would be increased by a factor of 6, with the main contribution coming from the square of the Ricci scalar. If there were no Yang-Mills fluxes on the boundary ρ would then be $\simeq 0.1104$, for which (37) and (38) give $\xi_1 \simeq 0.725$, $k \simeq -32.75$, $p_1 \simeq 0.430$, $A \simeq 0.699$, $q_1 \simeq -0.319$, and $b_1 \simeq 0.989B \simeq 0.28\kappa_{11}^{2/9}$. This is not really within the region where working to first order in p and q is justified.

The Einstein action on the 4 extended dimensions has the form:

$$S_{\text{Ein}} = \frac{1}{16\pi G_N} \int d^4\tilde{x} \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} R_{\mu\nu}(\tilde{g}), \quad (39)$$

where $\tilde{g}_{\mu\nu}$ differs from $\eta_{\mu\nu}$ by a small perturbation, that depends on the coordinates \tilde{x}^σ on the 4 extended dimensions, but not on the coordinates \bar{x}^A on \bar{H}^7 . G_N is Newton's constant, with the value [63]:

$$G_N = 6.7087 \times 10^{-33} \text{TeV}^{-2}, \quad (40)$$

so that $\sqrt{G_N} = 8.1907 \times 10^{-17} \text{TeV}^{-1} = 1.6160 \times 10^{-35}$ metres. Comparing with (3) and (14), and noting that $R_{IJ}{}^K{}_L$ and hence R_{IJ} are unaltered by rescaling the metric by a constant factor, so that $\sqrt{-G}G^{\mu\nu}R_{\mu\nu}(G) = A^4B^7\sqrt{-\tilde{g}}\sqrt{\tilde{g}}\frac{1}{A^2}\tilde{g}^{\mu\nu}R_{\mu\nu}(\tilde{g})$ everywhere on \bar{H}^7 except in the immediate vicinity of the HW boundary, where G_{IJ} here represents the metric obtained from (14) by replacing $\eta_{\mu\nu}$ by $\tilde{g}_{\mu\nu}$, we find, in the approximation of neglecting the volume of the region where $a(y)$ in (27) differs appreciably from A , that the intrinsic volume $\bar{V}_7 \equiv \int_{\bar{H}^7} d^7\bar{x} \sqrt{\bar{g}}$ of \bar{H}^7 is given by [1, 2, 3, 64, 12]:

$$\frac{A^2B^7\bar{V}_7}{2\kappa_{11}^2} = \frac{1}{16\pi G_N} \quad (41)$$

The experimental limits on the gravitational coupling constant in D dimensions are expressed in terms of a mass M_D , such that for $D = 11$, $M_{11} = (2\pi)^{7/9} \kappa_{11}^{-2/9} = 4.1764\kappa_{11}^{-2/9}$ [57, 63]. The latest limits from searches for virtual graviton exchange and graviton emission at the LHC [65, 66, 67, 68, 69, 70, 71, 72], and searches for microscopic black holes at the LHC [73, 74, 75], suggest that the experimental lower bound on M_{11} , for 7 flat extra dimensions, is now roughly $M_{11} \geq 2.3 \pm 0.7 \text{ TeV}$, corresponding to $\kappa_{11}^{-2/9} \geq 0.55 \pm 0.2 \text{ TeV}$.

From above, A is expected to lie in the range from about 0.7 to about 0.9, and the best value of B , determined by the PMS, is $B \simeq 0.28\kappa_{11}^{2/9}$. If $\kappa_{11}^{-2/9}$ was about 0.55 TeV,

so that B was around 0.51 TeV^{-1} , $A = 0.7$ would give $\bar{V}_7 \simeq 3.0 \times 10^{35}$, and $A = 0.9$ would give $\bar{V}_7 \simeq 1.8 \times 10^{35}$. Thus if \bar{H}^7 is reasonably isotropic, in the sense that it has an approximately spherical fundamental domain in 7-dimensional hyperbolic space H^7 , then from page 9 of [20], the current upper bound on the intrinsic diameter \bar{L}_7 of \bar{H}^7 is about 28, hence the current upper bound on the actual diameter L_7 of \bar{H}^7 is about $14 \text{ TeV}^{-1} \simeq 2.8 \times 10^{-18}$ metres.

From between (23) and (24) above, the intrinsic diameter \bar{L}_6 of the closed hyperbolic factor \bar{H}^6 of the HW boundary lies between about 5.7 and 6.0 if \bar{H}^6 is reasonably isotropic, so if both \bar{H}^7 and \bar{H}^6 are reasonably isotropic, the current upper bound on the ratio \bar{L}_7/\bar{L}_6 of their intrinsic diameters lies between about 4.9 and 4.7. And since the curvature radius b_1 of the HW boundary is $\simeq B$, this also gives the current upper bound on the ratio L_7/L_6 of their actual diameters.

Closed hyperbolic 7-manifolds $\bar{\bar{H}}^7$ of intrinsic volume $\bar{V}_7 \sim 10^{35}$ that have a closed hyperbolic minimal-area 6-cycle $\bar{\bar{H}}^6$ of intrinsic volume $\sim 10^6$, such that in suitable coordinates near $\bar{\bar{H}}^6$ the metric of sectional curvature -1 on $\bar{\bar{H}}^7$ has the form of the last two terms in (27) with $b = \cosh\left(\frac{y}{B}\right)$, might be relatively rare among $\bar{\bar{H}}^7$ with $\bar{V}_7 \sim 10^{35}$. For if cutting $\bar{\bar{H}}^7$ along $\bar{\bar{H}}^6$ separates $\bar{\bar{H}}^7$ into two connected components, let $\bar{\bar{H}}_{(2)}^7$ be formed by cutting $\bar{\bar{H}}^7$ along $y = 0$ and joining two copies of the larger volume component along this boundary, while if cutting $\bar{\bar{H}}^7$ along $\bar{\bar{H}}^6$ leaves $\bar{\bar{H}}^7$ connected, let $\bar{\bar{H}}_{(2)}^7$ be formed from two copies of $\bar{\bar{H}}^7$ cut along $y = 0$, by joining boundary b of copy 1 to boundary a of copy 2, and boundary b of copy 2 to boundary a of copy 1. Then the smallest non-zero intrinsic eigenvalue $\bar{\lambda}_1$ of the negative of the Laplace-Beltrami operator $\Delta \equiv \frac{1}{\sqrt{g}} \partial_A (\sqrt{g} g^{AB} \partial_B \cdot)$ on $\bar{\bar{H}}_{(2)}^7$ is bounded above by $\sim 10^{-29}$, since for any function $f(\bar{x})$ such that $\int_{\bar{\bar{H}}^7} \sqrt{g} f d^7 \bar{x} = 0$:

$$\bar{\lambda}_1 \leq \frac{\int_{\bar{\bar{H}}^7} \sqrt{g} g^{AB} (\partial_A f) (\partial_B f) d^7 \bar{x}}{\int_{\bar{\bar{H}}^7} \sqrt{g} f^2 d^7 \bar{x}}, \quad (42)$$

and we can choose f to be 1 on one of the two connected components of the manifold obtained from $\bar{\bar{H}}^7$ by deleting the region with $|y| < 1$, and -1 on the other such component, with a smooth transition across the region with $|y| < 1$ [76]. But from the discussion on pages 9 to 12 of [20], it seems possible that typical $\bar{\bar{H}}^n$, $n \geq 2$, of arbitrarily large intrinsic volume \bar{V}^n , will have few or no nonzero intrinsic eigenvalues $\bar{\lambda}$ of $-\Delta$ smaller than $\frac{(n-1)^2}{4}$ [12, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 16, 92, 93].

Closed $\bar{\bar{H}}^7$ that have a closed hyperbolic minimal-area 6-cycle $\bar{\bar{H}}^6$, such that in suitable coordinates the metric near $\bar{\bar{H}}^6$ is as above, exist with arbitrarily large values

of \bar{V}_7/\bar{V}_6 , for section 2.8.C of [94] gives examples for all $n \geq 2$ of \bar{H}^n that contain a 2-sided non-separating embedded closed hyperbolic hypersurface \bar{H}^{n-1} . If we take N copies of such an \bar{H}^n , cut each along that \bar{H}^{n-1} , and join side b of copy 1 to side a of copy 2, side b of copy 2 to side a of copy 3, \dots , and side b of copy N to side a of copy 1, we get a closed hyperbolic n -manifold $\bar{H}_{(N)}^n$ that is an N -fold cover of the original \bar{H}^n , so the ratio of the intrinsic volume $\bar{V}_{n(N)}$ of $\bar{H}_{(N)}^n$ to the intrinsic volume \bar{V}_{n-1} of that \bar{H}^{n-1} can be arbitrarily large. However these $\bar{H}_{(N)}^n$ are far from being reasonably isotropic for large N , because their intrinsic diameters and intrinsic volumes both grow in proportion to N , while from page 9 of [20], the intrinsic volume \bar{V}_n of a reasonably isotropic \bar{H}^n is approximately related to its intrinsic diameter \bar{L}_n , for large \bar{L}_n , by $\bar{V}_n \simeq \frac{S_{n-1}}{2^{n-1}(n-1)} e^{(n-1)\frac{\bar{L}_n}{2}}$, where S_{n-1} is the area of the unit $(n-1)$ -sphere.

2 The bosonic Kaluza-Klein modes of the super-gravity multiplet

I shall continue to work to leading order in the Lukas-Ovrut-Waldram (LOW) harmonic expansion of the energy-momentum tensor on \bar{H}^7 [21], and to assume that the vacuum fluxes are approximately uniformly distributed across \bar{H}^7 , so that the LOW expansion only needs to be applied over relatively small local regions of \bar{H}^7 . In addition to the assumption (1) on the vacuum flux bilinears, I shall assume that to leading order in the LOW harmonic expansion, expressions linear in the vacuum fluxes are zero. The Kaluza-Klein modes of the metric G_{IJ} and the 3-form gauge field C_{IJK} are then to a first approximation decoupled from each other, and can thus be treated separately. I shall use the convention stated between (8) and (9), that repeated lower coordinate indices are understood to be contracted with an inverse metric G^{IJ} .

2.1 The Kaluza-Klein modes of the 3-form gauge field

As stated just before (1), the vacuum 4-form fluxes are assumed to be proportional to harmonic 4-forms on \bar{H}^7 , and thus to solve the classical CJS field equations (13) for H_{IJKL} , and the quantum corrections to those field equations are neglected, so for a first approximation to the Kaluza-Klein modes of C_{IJK} , it is consistent to consider just the classical CJS action, whose bosonic part is (3). With the above assumptions on terms linear or bilinear in the vacuum fluxes, the vacuum fluxes do not affect the

Kaluza-Klein modes of C_{IJK} , and the $\epsilon_{11}CHH$ Chern-Simons term in (3) also plays no role. Thus for a first, classical, approximation to the masses of the Kaluza-Klein modes of C_{IJK} , we can neglect the vacuum fluxes completely, and consider just the $-\frac{1}{48}H_{IJKL}H_{IJKL}$ term in (3). Then after adding gauge-fixing terms as follows, and noting that for the metric ansatz (14), covariant derivatives D_μ are ordinary derivatives ∂_μ , and commute with each other and with D_A , we have:

$$\begin{aligned}
& -\frac{1}{48}H_{IJKL}H_{IJKL} - \frac{1}{4}(aD_\mu C_{\mu\nu\sigma} + \frac{1}{a}D_A C_{A\nu\sigma})(aD_\tau C_{\tau\nu\sigma} + \frac{1}{a}D_B C_{B\nu\sigma}) \\
& - \frac{1}{2}(bD_\mu C_{\mu A\sigma} + \frac{1}{b}D_B C_{BA\sigma})(bD_\nu C_{\nu A\sigma} + \frac{1}{b}D_E C_{EA\sigma}) \\
& - \frac{1}{4}(cD_\mu C_{\mu AB} + \frac{1}{c}D_E C_{EAB})(cD_\nu C_{\nu AB} + \frac{1}{c}D_F C_{FAB}) = \\
& = \frac{1}{12} \left\{ -\partial_\mu C_{\nu\sigma\tau} \partial_\mu C_{\nu\sigma\tau} + 3(1-a^2) \partial_\mu C_{\mu\sigma\tau} \partial_\nu C_{\nu\sigma\tau} - D_A C_{\mu\nu\sigma} D_A C_{\mu\nu\sigma} \right\} \\
& + \frac{1}{4} \left\{ -\partial_\mu C_{\nu\sigma A} \partial_\mu C_{\nu\sigma A} + 2(1-b^2) \partial_\mu C_{\mu\sigma A} \partial_\nu C_{\nu\sigma A} - D_A C_{\mu\nu B} D_A C_{\mu\nu B} \right. \\
& \quad \left. + D_A C_{\mu\nu B} D_B C_{\mu\nu A} - \frac{1}{a^2} D_A C_{\mu\nu A} D_B C_{\mu\nu B} \right\} \\
& + \frac{1}{4} \left\{ -\partial_\mu C_{\nu AB} \partial_\mu C_{\nu AB} + (1-c^2) \partial_\mu C_{\mu AB} \partial_\nu C_{\nu AB} - D_A C_{\mu BE} D_A C_{\mu BE} \right. \\
& \quad \left. + 2D_A C_{\mu BE} D_B C_{\mu AE} - \frac{2}{b^2} D_A C_{\mu AE} D_B C_{\mu BE} \right\} + \frac{1}{12} \left\{ -\partial_\mu C_{ABE} \partial_\mu C_{ABE} \right. \\
& \quad \left. - D_A C_{BEF} D_A C_{BEF} + 3D_A C_{BEF} D_B C_{AEF} - \frac{3}{c^2} D_A C_{AEF} D_B C_{BEF} \right\},
\end{aligned} \tag{43}$$

where a , b , and c are gauge parameters. Derivatives in the right-hand side of (43) act only on the smallest object to their immediate right. C_{IJK} and H_{IJKL} in (43) refer to the Kaluza-Klein modes only. The modes of different spin along the extended dimensions are decoupled in the right-hand side of (43). If we choose $a = b = c = 1$, which corresponds to a gauge-fixing term $-\frac{1}{4}D_I C_{IKL} D_J C_{JKL}$ and is effectively Feynman gauge, then after making a Kaluza-Klein ansatz such as $C_{\mu\nu A} = c_{\mu\nu}(\tilde{x}) \omega_A(\bar{x})$ in the corresponding field equations and separating the field equations, the field equations on \bar{H}^7 in the metric \bar{g}_{AB} of sectional curvature -1 have the form $-(\delta d + d\delta)\omega = \bar{m}^2\omega$, where $\delta d + d\delta$ is the Hodge - de Rham Laplacian, so the intrinsic masses \bar{m} of the modes with p A -type indices, $0 \leq p \leq 3$, are given by the spectrum of the negative of the Hodge - de Rham Laplacian for p -forms on \bar{H}^7 . From pages 42 to 43 of [20], this

means that their masses, as seen on the HW boundary, are $m = \frac{A}{B}\bar{m}$, where A and B are the constants in the metric ansatz (14).

Choosing alternatively now the limiting gauge choice $a \rightarrow 0$, $b \rightarrow 0$, $c \rightarrow 0$, we obtain Proca-type unitary gauges for the massive antisymmetric tensor fields on the extended dimensions [95, 96], and Landau-gauge-like restrictions such as $D_A C_{\mu\nu A} = 0$ on the dependence of the modes on position on \bar{H}^7 , which means that some of the massive modes obtained in Feynman gauge are unphysical, and would be cancelled by corresponding Faddeev-Popov ghosts in Feynman gauge.

From pages 9 to 12 and 16 to 17 of [20], it seems likely that classically, the lightest massive modes of a p -form gauge field on \bar{H}^7 , for $p < 3$, will have intrinsic mass $\bar{m} = 3 - p$ [12, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 16, 92, 93]. From pages 42 to 43 of [20], this means that their mass, as seen on the HW boundary, is $m = (3 - p) \frac{A}{B}$.

2.1.1 The classically massless harmonic 3-form modes

In addition to the classically massive modes of C_{IJK} , there are classically massless modes $C_{ABC} = C(\tilde{x})\omega_{ABC}(\bar{x})$, $C_{\mu AB} = C_\mu(\tilde{x})\omega_{AB}(\bar{x})$, and $C_{\mu\nu A} = C_{\mu\nu}(\tilde{x})\omega_A(\bar{x})$, corresponding respectively to harmonic 3-forms $\omega_{ABC}(\bar{x})$, 2-forms $\omega_{AB}(\bar{x})$, and 1-forms $\omega_A(\bar{x})$, on \bar{H}^7 . The field strengths H_{ABCD} , $H_{\mu ABC}$, and $H_{\mu\nu AB}$, that would occur respectively in their classical mass terms, vanish identically, so they can only obtain masses from quantum corrections that arise from interaction terms that contain C_{IJK} explicitly, so the relevant terms in (2) are the CJS Chern-Simons term $\epsilon_{11}CHH$ in (3), and the Green-Schwarz term (10) in (4). The harmonic 0-form mode $C_{\mu\nu\sigma}(\tilde{x})$ is classically a pure gauge mode, with no physical degrees of freedom.

If these modes all acquire masses $\sim \kappa_{11}^{-2/9}$ from quantum corrections, then only the harmonic 3-form modes C_{ABC} are expected to be sufficiently numerous for their large number to compensate for the gravitational suppression of their couplings enough for them to be seen at the LHC, because from pages 17 to 19 of [20], the Betti number B_3 of \bar{H}^7 is estimated as $\sim \frac{\bar{V}_7}{\ln \bar{V}_7}$, while the Betti numbers B_2 and B_1 of \bar{H}^7 are estimated as powers strictly less than 1 of \bar{V}_7 [97, 98, 99, 100, 91, 101].

The leading contribution to the squared masses of the harmonic 3-form modes of C_{ABC} , in the presence of the fluxes H_{ABCD} proportional to harmonic 4-forms on \bar{H}^7 , arises from the CJS Chern-Simons term in (3), on integrating out $H_{\mu\nu\sigma\tau}$ [102]. The $\mu\nu\sigma\tau\rho$ component of the Bianchi identity for H_{IJKL} is satisfied automatically,

so after making a suitable choice of gauge for $C_{\mu\nu\sigma}$, we can change variables from $C_{\mu\nu\sigma}$ to $H_{\mu\nu\sigma\tau}$, which is now an unconstrained scalar multiple of $\epsilon_{\mu\nu\sigma\tau}$. In particular, choosing the Lorentz gauge condition $\partial^\mu C_{\mu\nu\sigma} = 0$, and suitable boundary conditions as $x^0 = \tilde{x}^0 \rightarrow \pm\infty$, we can write:

$$C_{\mu\nu\sigma}(\tilde{x}, \bar{x}) = \int d^4\tilde{x}' \frac{\partial}{\partial \tilde{x}_\rho} G_{4F}(\tilde{x} - \tilde{x}') H_{\mu\nu\sigma\rho}(\tilde{x}', \bar{x}), \quad (44)$$

where $G_{4F}(\tilde{x} - \tilde{x}')$ is the Feynman propagator for a massless scalar in 3+1 dimensions, which satisfies $-\frac{\partial^2}{\partial \tilde{x}^\mu \partial \tilde{x}_\mu} G_{4F}(\tilde{x} - \tilde{x}') = \delta^4(\tilde{x} - \tilde{x}')$.

Neglecting the leading quantum correction $\Gamma_{\text{SG}}^{(8,\text{bos})}$, we can now integrate out $H_{\mu\nu\sigma\tau}$, since it occurs quadratically in the CJS action (3). The terms containing $H_{\mu\nu\sigma\tau}$ quadratically are the $-H_{\mu\nu\sigma\tau} H^{\mu\nu\sigma\tau}$ and $-4H_{\mu\nu\sigma A} H^{\mu\nu\sigma A}$ terms from $-H_{IJKL} H^{IJKL}$. The $-H_{\mu\nu\sigma\tau} H^{\mu\nu\sigma\tau}$ term corresponds to a multiple of the identity matrix in the $H_{\mu\nu\sigma\tau}$ Hilbert space, and we can expand the inverse of the matrix defining the quadratic form corresponding to these two terms as a power series in the matrix corresponding to the $-4\partial_A C_{\mu\nu\sigma} \partial^A C^{\mu\nu\sigma}$ part of the second term.

However when we evaluate the expectation value of the resulting mass term in a specific classical 3-form mode on \bar{H}^7 , each derivative ∂_A , (which as it occurs here is a covariant derivative for the metric (14), since the only non-vanishing Christoffel symbols are $\Gamma_A^B C^B$), will roughly give either a factor of the classical mass of that mode, which is zero for the harmonic 3-form modes, or a factor of $\frac{1}{\bar{B}}$. The harmonic 3-forms are the most covariantly smooth 3-form modes, so I shall assume that for them, any such factor of $\frac{1}{\bar{B}}$ is accompanied by a factor $\frac{1}{\bar{L}_7}$, where \bar{L}_7 , the intrinsic diameter of \bar{H}^7 , is ~ 27 for $\bar{V}_7 \sim 10^{34}$, if \bar{H}^7 is reasonably isotropic, in the sense that it has a fundamental domain in H^7 that is approximately spherical. So for a first approximation to the mass of the harmonic 3-form modes, I shall neglect the $-\partial_A C_{\mu\nu\sigma}$ term in $H_{\mu\nu\sigma A}$.

To extract the relevant part of the Chern-Simons term in (3), we split each index $I_1 \dots I_{11}$ independently into its μ range and its A range, and look for terms that can produce $H_{\mu\nu\sigma\tau}$, after integrations by parts if necessary. To get a $C_{\mu\nu\sigma}$, one of the three factors $C_{I_1 I_2 I_3} H_{I_4 \dots I_7} H_{I_8 \dots I_{11}}$ has to have at least 3 μ -type indices.

There are 2 terms like $\epsilon_{(11)}^{ABCDEFG\mu\nu\sigma\tau} C_{ABC} H_{DEFG} H_{\mu\nu\sigma\tau}$.

There are 8 terms like $\epsilon_{(11)}^{\mu\nu\sigma\tau ABCDEFG} C_{\mu\nu\sigma} H_{\tau ABC} H_{DEFG}$, which contains a term that on integration by parts, gives $\frac{1}{4} \epsilon_{(11)}^{ABCDEFG\mu\nu\sigma\tau} C_{ABC} H_{DEFG} H_{\mu\nu\sigma\tau}$.

There are 32 terms like $\epsilon_{(11)}^{ABC\mu\nu\sigma D\tau EFG} C_{ABC} H_{\mu\nu\sigma D} H_{\tau EFG}$, which contains a term that on integration by parts, gives

$$-\frac{1}{4}\epsilon_{(11)}^{ABC\mu\nu\sigma D\tau EFG}C_{ABC}(\partial_D H_{\mu\nu\sigma\tau})C_{EFG}=0.$$

Thus the relevant terms in (3) containing $H_{\mu\nu\sigma\tau}$ are:

$$\frac{1}{96\kappa_{11}^2}\int_{\mathcal{B}}d^{11}xe\left(-H_{\mu\nu\sigma\tau}H^{\mu\nu\sigma\tau}-\frac{1}{108}\epsilon_{(11)}^{ABCDEFG\mu\nu\sigma\tau}C_{ABC}H_{DEFG}H_{\mu\nu\sigma\tau}\right). \quad (45)$$

After completing the square and integrating out $H_{\mu\nu\sigma\tau}$, this becomes:

$$-\frac{1}{96\kappa_{11}^2}\int_{\mathcal{B}}d^{11}xe\frac{4}{6^5}\epsilon_{(7)}^{ABCDEFG}\epsilon_{(7)HIJKLMN}C_{ABC}H_{DEFG}C^{HIJ}H^{KLMN}. \quad (46)$$

I shall now assume that to leading order in the LOW harmonic expansion, the vacuum fluxes satisfy:

$$H_{ABCD}H^{EFGH}=\frac{6h^2}{5B^8}\delta_A^E\delta_B^F\delta_C^G\delta_D^H=\frac{h^2}{120B^8}\epsilon_{(7)ABCDIJK}\epsilon_{(7)}^{EFGHIJK}, \quad (47)$$

where the coefficient is fixed by (1), and as with (1), the LOW expansion only needs to be applied over relatively small local regions of \bar{H}^7 , due to the approximately uniform distribution of the fluxes across \bar{H}^7 . All indices in (46) and (47) are tangential to \bar{H}^7 . After adding the kinetic term $-4H_{\mu ABC}H^{\mu ABC}$ from $-H_{IJKL}H^{IJKL}$, (46) becomes:

$$\frac{1}{96\kappa_{11}^2}\int_{\mathcal{B}}d^{11}xe\left(-4H_{\mu ABC}H^{\mu ABC}-\frac{4h^2}{45B^8}C^{ABC}C_{ABC}\right). \quad (48)$$

Thus within the above approximations, all the harmonic 3-form modes of C_{ABC} obtain the same intrinsic mass $\bar{m}=\frac{h}{\sqrt{45}B^3}$. From subsection 1.1 above, the best value of $\eta=\frac{h}{B^3}$ chosen by the PMS is $\eta\simeq 1.425$, so $\bar{m}\simeq 0.2$. Thus the mass of these modes, as seen on the HW boundary, is $m\simeq 0.2\frac{A}{B}$.

2.1.2 The coupling of the harmonic 3-form modes to the SM gauge bosons

The coupling of the harmonic 3-form modes of C_{ABC} to the SM fields can be obtained by integrating out $H_{\mu\nu\sigma\tau}$, in the same way as was done above to calculate their mass. In Moss's improved form of Hořava-Witten theory, the boundary condition for H_{IJKL} has the form [8, 9, 10, 11]:

$$H_{UVWX}|_{y=y_{1+}}=\frac{1}{2\pi}\left(\frac{\kappa_{11}}{4\pi}\right)^{2/3}\left(-3F_{[UV}^A F_{WX]}^A+\bar{\chi}^A\Gamma_{[UVW}(D_X)\chi)^A\right)+\dots, \quad (49)$$

where χ^A is the gaugino, and \dots denotes terms that involve the gravitino or R_{IJKL} or H_{IJKL} . This can be integrated to:

$$C_{UVW}|_{y=y_{1+}}=\frac{1}{4\pi}\left(\frac{\kappa_{11}}{4\pi}\right)^{2/3}\left(-\frac{1}{30}\Omega_{UVW}^{(Y)}+\frac{1}{4}\bar{\chi}^A\Gamma_{UVW}\chi^A\right)+\lambda_{UVW}+\dots, \quad (50)$$

where

$$\Omega_{UVW}^{(Y)} = 6\text{tr} \left(A_{[U} \partial_V A_{W]} + \frac{2}{3} i A_{[U} A_V A_{W]} \right) \quad (51)$$

is the Yang-Mills Chern-Simons 3-form, λ_{UVW} is an arbitrary closed 3-form on the boundary, and \dots denotes terms that involve the gravitino or the Lorentz Chern-Simons 3-form or H_{IJKL} .

With the notation of (27) above, let $\tilde{C}_{UVW}(\tilde{x}, \hat{x})$ denote the right-hand side of (50). If we neglect $H_{\mu\nu\sigma\tau}$ and $H_{\mu\nu\sigma a}$ and the CJS Chern-Simons term, the CJS field equations (13) for H_{IJKL} include an equation $\frac{\partial}{\partial y} (a^4 b^6 a^{-6} H_{\mu\nu\sigma y}) = 0$, whose solution is $H_{\mu\nu\sigma y} = \frac{a^2}{b^6} f_{\mu\nu\sigma}(\tilde{x}, \hat{x})$. If we further neglect $C_{\mu\nu y}$, we then find that the form of $C_{\mu\nu\sigma}$ induced by $\tilde{C}_{\mu\nu\sigma}(\tilde{x}, \hat{x})$ is $C_{\mu\nu\sigma} = f_{\mu\nu\sigma}(\tilde{x}, \hat{x}) \int_y^\infty \frac{a^2(y')}{b^6(y')} dy'$. In the approximation of neglecting the perturbations $p(y)$ and $q(y)$ defined between (27) and (28) above, so that $a(y) = A$, and $b(y) = B \cosh \frac{y}{B}$, we have:

$$\int_y^\infty \frac{a^2(y')}{b^6(y')} dy' \simeq \int_y^\infty \frac{A^2}{B^6 \left(\cosh \frac{y}{B} \right)^6} = \frac{A^2 e^{-\frac{5y}{B}} \left(10e^{\frac{4y}{B}} + 5e^{\frac{2y}{B}} + 1 \right)}{30B^5 \left(\cosh \frac{y}{B} \right)^5}, \quad (52)$$

so in the further approximation of setting y_1 , the value of y at the boundary, to 0, the flux $H_{\mu\nu\sigma y}^{(\text{ind})}$ induced by $\tilde{C}_{\mu\nu\sigma}(\tilde{x}, \hat{x})$ is:

$$H_{\mu\nu\sigma y}^{(\text{ind})}(\tilde{x}, \hat{x}, y) \simeq \frac{15B^5 a^2}{8A^2 b^6} \tilde{C}_{\mu\nu\sigma}(\tilde{x}, \hat{x}). \quad (53)$$

The principal coupling between the SM gauge bosons and the C_{ABC} modes arises from the cross term between $H_{\mu\nu\sigma y}^{(\text{ind})}$, and the flux $H_{\mu\nu\sigma y}^{(\text{spont})}$ that originates from (44), in $-\frac{4}{96\kappa_{11}^2} \int_{\mathcal{B}} d^{11}x e (H_{\mu\nu\sigma y} H^{\mu\nu\sigma y})$, which is one of the $H_{\mu\nu\sigma A} H^{\mu\nu\sigma A}$ terms neglected in deriving (45). Using the algebraic field equation for $H_{\mu\nu\sigma\tau}$ that follows from (45), we find:

$$H_{\mu\nu\sigma y}^{(\text{spont})}(\tilde{x}, \hat{x}, y) = \frac{1}{216} \frac{\partial}{\partial y} \left(\epsilon_{(11)}^{abcdefy}{}_{\mu\nu\sigma\rho} \int d^4\tilde{x}' \frac{\partial}{\partial \tilde{x}'_\rho} G_{4F}(\tilde{x} - \tilde{x}') (3C_{aby}(\tilde{x}', \hat{x}, y) H_{cdef} + 4C_{abc}(\tilde{x}', \hat{x}, y) H_{defy}) \right), \quad (54)$$

where H_{cdef} and H_{defy} are the vacuum fluxes that to leading order in the LOW harmonic expansion, applied over relatively small local regions of \bar{H}^7 , satisfy (1) and (47). So considering just the Yang-Mills term in $\tilde{C}_{\mu\nu\sigma}(\tilde{x}, \hat{x})$, the principal coupling between

the SM gauge bosons and the C_{ABC} modes is:

$$\begin{aligned}
& -\frac{8}{96\kappa_{11}^2} \int_{\mathcal{B}} d^{11}x e H_{\mu\nu\sigma y}^{(\text{ind, YM})} H^{(\text{spont})\mu\nu\sigma y} \\
& \simeq \frac{5B^5}{2^{11}3^2\pi A^2\kappa_{11}^2} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \int d^4\tilde{x} \int_{\bar{H}^6} d^6\hat{x} \sqrt{\hat{g}} \epsilon_{(4)}^{\mu\nu\sigma\rho} F_{[\mu\nu}^{\mathcal{A}} F_{\sigma\rho]}^{\mathcal{A}} \int_{y_1}^{\infty} dy a^6 \frac{\partial}{\partial y} \int d^4\tilde{x}' G_{4F}(\tilde{x} \\
& \quad - \tilde{x}') \epsilon_{(7)}^{abcdefy} (3C_{aby}(\tilde{x}', \hat{x}, y) H_{cdef} + 4C_{abc}(\tilde{x}', \hat{x}, y) H_{defy}).
\end{aligned} \tag{55}$$

The integral is strongly localized near the boundary, because $\epsilon_{(7)}^{abcdefy}$ is b^{-6} times ± 1 or 0, and $a \rightarrow A$ and $b \rightarrow B \cosh \frac{y}{B}$ as $y \rightarrow \infty$. If we again neglect the perturbation $p(y)$, so that $a(y) = A$, and neglect the massive Kaluza-Klein modes of the Yang-Mills gauge fields, then the coupling is approximately:

$$\begin{aligned}
& -\frac{5A^4B^5}{2^{11}3^2\pi\kappa_{11}^2} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \sum_{(n)} \int d^4\tilde{x} \epsilon_{(4)}^{\mu\nu\sigma\rho} F_{[\mu\nu}^{\mathcal{A}} F_{\sigma\rho]}^{\mathcal{A}} \int d^4\tilde{x}' G_{4F}(\tilde{x} - \tilde{x}') \\
& \quad \times C_{(n)}(\tilde{x}') \int_{\bar{H}^6} d^6\hat{x} \sqrt{\hat{g}} \epsilon_{(7)}^{ABCDEFG} \omega_{(n)ABC}(\hat{x}, y_1) H_{DEFG}(\hat{x}, y_1),
\end{aligned} \tag{56}$$

where $C_{ABC}(\tilde{x}, \hat{x}, y)$ has been expanded in mass eigenmodes as:

$$C_{ABC}(\tilde{x}, \hat{x}, y) = \sum_{(n)} C_{(n)}(\tilde{x}) \omega_{(n)ABC}(\hat{x}, y). \tag{57}$$

If we restrict this sum to the harmonic 3-form modes, with mass $m \simeq 0.2 \frac{A}{B}$, then when the coupling (56) is inserted into a momentum-space Feynman diagram for two gluons to turn into a $C_{(n)}$, which then decays to 2 or 3 SM gauge bosons, the massless propagator G_{4F} at each end of the $C_{(n)}$ propagator becomes a factor $\frac{1}{m^2}$ near the $C_{(n)}$ mass shell. Thus near the $C_{(n)}$ mass shell, the coupling (56) has the standard form for the coupling of the SM gauge bosons to axion fields $C_{(n)}(\tilde{x})$ [102]. However the $C_{(n)}$ fields, whose mass would be around a TeV, are very different from conventional axions, which are extremely light [103, 104].

If candidates for the $C_{(n)}$ modes are observed and their decays to 3 gluon jets can be identified, the coupling (56) could be tested by plotting the energies of the 3 gluon jets, in the reconstructed rest frame of the candidate $C_{(n)}$ mode, on a Dalitz plot [105, 106, 107]. The coupling is proportional to the 4-momentum of the $C_{(n)}$ mode because $\epsilon_{(4)}^{\mu\nu\sigma\rho} F_{[\mu\nu}^{\mathcal{A}} F_{\sigma\rho]}^{\mathcal{A}}$ is a total derivative, so in radiation gauge in the rest frame of the $C_{(n)}$ mode, the polarizations of the gluons in the 3-gluon term in (51) must be linearly

independent, and that is not possible if the 3 gluons are collinear. Thus the amplitude vanishes for 3 collinear gluons, which means that the distribution of events will be depleted near the edges of the Dalitz plot, which are the lines $2E_1 = m$, $2E_2 = m$, and $2E_3 = m$. The background to this effect would include both the QCD background, and the decays of the candidate $C_{(n)}$ modes to 2 gluons, where one of the gluons radiates a third gluon.

2.2 The Kaluza-Klein modes of the metric

The background solution of the field equation for the metric depends essentially on the presence of the leading quantum correction (4) to the CJS action, so the presence of that term has to be taken into account in studying the Kaluza-Klein modes of the metric. However (4) is 8th order in derivatives, so it has to be treated as a perturbation of the momentum-dependent terms in the action of the Kaluza-Klein modes. For a first approximation, I shall neglect the contribution of (4) to the momentum-dependent terms in the action of the Kaluza-Klein modes, and calculate the contribution of (4) to the mass squared of the dilaton/radion. I shall then assume that the contribution of (4) to the mass squared of the other light Kaluza-Klein modes of the metric is of similar magnitude to its contribution to the mass squared of the dilaton/radion.

I shall write the perturbed metric as $\bar{\bar{G}}_{IJ} \equiv G_{IJ} + 2h_{IJ}$, where G_{IJ} is the metric defined by (14), and h_{IJ} is the perturbation tensor. Indices will still be raised and lowered with G_{IJ} , and covariant derivatives D_I will still be defined in terms of the unperturbed metric G_{IJ} , and satisfy $D_I G_{JK} \equiv 0$. Repeated lower coordinate indices are still understood to be contracted with G^{IJ} . The inverse of $\bar{\bar{G}}_{IJ}$ will be written as $\check{\check{G}}^{IJ} \equiv G^{IJ} - 2h^{IJ} + 4h^I_K h^{KJ} - \dots$, to distinguish it from the tensor obtained from $\bar{\bar{G}}_{IJ}$ by raising both its indices with G^{IJ} . Other quantities defined in terms of $\bar{\bar{G}}_{IJ}$ will be written with a double bar above them. Repeated lower coordinate indices are still understood to be contracted with an unperturbed inverse metric G^{IJ} , in accordance with the convention stated between (8) and (9). Then we have the tensor:

$$\Delta_I^J{}_K \equiv \bar{\bar{\Gamma}}_I^J{}_K - \Gamma_I^J{}_K = \check{\check{G}}^{JL} (D_I h_{KL} + D_K h_{IL} - D_L h_{IK}), \quad (58)$$

and the tensor:

$$\bar{\bar{R}}_{IJ}{}^K{}_L = R_{IJ}{}^K{}_L + D_I \Delta_J^K{}_L - D_J \Delta_I^K{}_L + \Delta_I^K{}_M \Delta_J^M{}_L - \Delta_J^K{}_M \Delta_I^M{}_L, \quad (59)$$

and also:

$$\bar{e} = \sqrt{-\bar{G}} = e \left(1 + h_{II} - h_{IJ}h_{JI} + \frac{1}{2}h_{II}h_{JJ} \right), \quad (60)$$

where the double-barred quantities in the above equations are defined in terms of \bar{G}_{IJ} with their indices in the positions shown. We then find [108, 109, 110]:

$$\begin{aligned} \bar{e}\bar{R} &= \bar{e}\check{\bar{G}}^{IJ}\bar{R}_{IJ} = \bar{e}\check{\bar{G}}^{IJ}\bar{R}_{KI}{}^K{}_J \\ &\cong e \left(R + h_{II}R - 2h_{IJ}R_{IJ} + 2h_{IK}h_{KJ}R_{IJ} + 2h_{IK}h_{JL}R_{IJKL} - 2h_{KK}h_{IJ}R_{IJ} - h_{IJ}h_{JI}R \right. \\ &\quad \left. + \frac{1}{2}h_{II}h_{JJ}R - D_K h_{IJ}D_K h_{IJ} + 2D_I h_{IK}D_J h_{JK} - 2D_I h_{IJ}D_J h_{KK} + D_K h_{II}D_K h_{JJ} \right), \end{aligned} \quad (61)$$

where \cong means up to the addition of total derivative terms, and the identity:

$$eD_I h_{JK}D_J h_{IK} \cong eD_I h_{IK}D_J h_{JK} - eh_{IK}h_{KJ}R_{IJ} + eh_{IK}h_{JL}R_{IJKL} \quad (62)$$

has been used.

Let X_8 be defined such that $\Gamma_{\text{SG}}^{(8,\text{bos})}$, in (4), is $\frac{1}{2\kappa_{11}^2} \int_{\mathcal{B}} d^{11}x e X_8$. Let $\bar{\bar{X}}_8$ denote X_8 as calculated from (4), (6), (9), (10), (11), and the definition of \tilde{Z} as explained after (11), with G_{IJ} replaced by $\bar{G}_{IJ} = G_{IJ} + 2h_{IJ}$, and let $[\bar{\bar{X}}_8]_{\text{w.o.}\Delta}$ denote $\bar{\bar{X}}_8$, but with $\Delta_I{}^J{}_K$ in (59) set to 0, so that $[\bar{\bar{X}}_8]_{\text{w.o.}\Delta}$ does not contain any derivatives acting on h_{IJ} . Let $\left. \frac{\partial[\bar{\bar{X}}_8]_{\text{w.o.}\Delta}}{\partial h_{IJ}} \right|_G$ denote the ordinary derivative of $[\bar{\bar{X}}_8]_{\text{w.o.}\Delta}$ with respect to h_{IJ} at $h_{IJ} = 0$, where as usual in differentiating a function that depends on a symmetric tensor, all components of the tensor are treated as independent in the argument of the function, so that $\left. \frac{\partial[\bar{\bar{X}}_8]_{\text{w.o.}\Delta}}{\partial h_{IJ}} \right|_G h_{IJ}$ is the linear term in the expansion of $[\bar{\bar{X}}_8]_{\text{w.o.}\Delta}$ in powers of h_{IJ} . Let $[\bar{\bar{X}}_8]_{1\Delta}$ denote the part of the linear term in the expansion of $\bar{\bar{X}}_8$ in powers of h_{IJ} that arises from $\Delta_I{}^J{}_K$ *only*, and let $[\bar{\bar{X}}_8]_{2,\text{rd}}$ denote the quadratic term in the expansion of $\bar{\bar{X}}_8$ in powers of h_{IJ} , but with the term $-2\left. \frac{\partial[\bar{\bar{X}}_8]_{\text{w.o.}\Delta}}{\partial h_{IJ}} \right|_G h_{IK}h_{KJ}$, that originates from the quadratic term in the expansion of $\check{\bar{G}}^{IJ}$, removed. Let $\Lambda_{\text{SG}}^{(\text{bos})}$ be the integrand of $\Gamma_{\text{SG}}^{(\text{bos})}$ in (2). Then the expansion of $2\kappa_{11}^2 \bar{\bar{\Lambda}}_{\text{SG}}^{(\text{bos})}$ without the metric-independent CJS Chern-Simons term, through quadratic order in h_{IJ} , up to total derivative terms, can

be written:

$$\begin{aligned}
& \bar{e}\bar{R} - \frac{1}{48}\bar{e}\bar{G}^{IM}\bar{G}^{JN}\bar{G}^{KO}\bar{G}^{LP}H_{IJKL}H_{MNOP} + \bar{e}\bar{X}_8 \\
& \cong e\left(1 + h_{NN} - h_{MN}h_{NM} + \frac{1}{2}h_{MM}h_{NN}\right)\left\{R - \frac{1}{48}H_{IJKL}H_{IJKL} + X_8\right\} \\
& + e(1 + h_{NN})(-2h_{IM} + 4h_{IO}h_{OM})\left\{R_{IM} - \frac{1}{12}H_{IJKL}H_{MJKL} - \frac{1}{2}\frac{\partial[\bar{X}_8]_{\text{w.o.}\Delta}}{\partial h_{IM}}\bigg|_G\right\} \\
& - eD_Ih_{JK}D_Ih_{JK} + 2eD_Ih_{IK}D_Jh_{JK} - 2eD_Ih_{IJ}D_Jh_{KK} \\
& + eD_Ih_{JJ}D_Ih_{KK} - 2eh_{IK}h_{KJ}R_{IJ} + 2eh_{IK}h_{JL}R_{IJKL} \\
& - \frac{1}{2}eh_{IM}h_{JN}H_{IJKL}H_{MNKL} + eh_{NN}[\bar{X}_8]_{1\Delta} + e[\bar{X}_8]_{2,\text{rd.}}
\end{aligned} \tag{63}$$

For the unperturbed solution considered here, the field equation (18) resulting from varying A , or equivalently, the field equation resulting from varying $G_{\mu\nu}$, states that the action is zero, so the contents of the first pair of braces in (63) are 0. This requires fine-tuning the root mean square flux strength h in (1), and from [111] or page 34 of [20], this can easily be achieved to the required precision of about 1 part in 10^{90} of h^2 , due to the large flux numbers of the fluxes wrapping typical 4-cycles of \bar{H}^7 , with intrinsic 4-area $\sim 10^{30}$, even when the fine-tuning is required to hold over the relatively small local regions over which the Lukas-Ovrut-Waldram harmonic expansion [21] is assumed to be applied. The observed cosmological vacuum energy density of about $(2.3 \times 10^{-3} \text{ eV})^4$ [112, 113, 114] is negligible for terrestrial laboratory experiments, and I shall here treat the contents of the first pair of braces in (63) as exactly 0. The contents of the second pair of braces in (63) are then also exactly 0 in consequence of the field equation (19) resulting from varying B , or equivalently, the field equation resulting from varying G_{AB} , since to first order in h_{IJ} , the only h_{IJ} -dependent terms in the right-hand side of (59) are the two $D\Delta$ terms, and when we integrate the D away from the Δ by parts in $\bar{\Gamma}_{\text{SG}}^{(8,\text{bos})}$, the D can only act on an R_{IJKL} , whose covariant derivatives are all 0, by the local symmetry of the metric ansatz (14).

When we split the index I to μ and A , the third and fourth terms after the second pair of braces in (63) contain mixing terms between $h_{\mu\nu}$ and the dilaton/radion, which is here proportional to h_{AA} . This mixing arises because the coefficient of $\sqrt{-\bar{g}}R(\bar{g})$ in (39), when (39) is derived from (3) by integration over \bar{H}^7 , is proportional to the volume \bar{V}_7 of \bar{H}^7 . This mixing can always be removed in a manner consistent with general covariance along the extended dimensions, by making a dilaton-dependent conformal

transformation of the metric $\check{g}_{\mu\nu}$ along the extended dimensions, of the form $\check{g}_{\mu\nu} = \left(\frac{\bar{V}_7}{V_7}\right)^{-\frac{2}{d-2}} \check{g}_{\mu\nu}$, where d is the number of extended dimensions, here 4. This is usually referred to as going to Einstein frame [115].

To the relevant order for the mixing terms, $\frac{\bar{V}_7}{V_7} = 1 + h_{AA}$, from (60). Thus from $\check{g}_{\mu\nu} = \bar{G}_{\mu\nu} = G_{\mu\nu} + 2h_{\mu\nu}$, and defining $\check{\check{g}}_{\mu\nu} \equiv G_{\mu\nu} + 2s_{\mu\nu}$, we have $h_{\mu\nu} = s_{\mu\nu} - \frac{1}{d-2}h_{AA}G_{\mu\nu}$. We also define $t_{AB} \equiv h_{AB} - \frac{1}{n}h_{CC}G_{AB}$, where n is the number of compact dimensions, here 7, so that $t_{AA} = 0$. Then after adding gauge-fixing terms as follows, (63) becomes:

$$\begin{aligned}
& -eD_I h_{JK} D_I h_{JK} + 2eD_I h_{IK} D_J h_{JK} - 2eD_I h_{IJ} D_J h_{KK} \\
& + eD_I h_{JJ} D_I h_{KK} - 2eh_{IK} h_{KJ} R_{IJ} + 2eh_{IK} h_{JL} R_{IJKL} \\
& - \frac{1}{2} eh_{IM} h_{JN} H_{IJKL} H_{MNKL} + eh_{NN} [\bar{X}_8]_{1\Delta} + e[\bar{X}_8]_{2,\text{rd.}} \\
& - 2e \left(\tilde{a} D_\mu s_{\mu\nu} + \tilde{b} D_\nu s_{\mu\mu} + \frac{1}{\tilde{a}} D_A h_{\nu A} \right) \left(\tilde{a} D_\sigma s_{\sigma\nu} + \tilde{b} D_\nu s_{\sigma\sigma} + \frac{1}{\tilde{a}} D_B h_{\nu B} \right) \\
& - e \frac{\tilde{a} + \tilde{b}}{\tilde{a}} \left(-2 D_\mu h_{\mu B} + D_B s_{\mu\mu} - \frac{\tilde{a}}{\tilde{a} + \tilde{b}} \left(D_A h_{AB} + \frac{1}{d-2} D_B h_{AA} \right) \right) \left(-2 D_\nu h_{\nu B} \right. \\
& \left. + D_B s_{\nu\nu} - \frac{\tilde{a}}{\tilde{a} + \tilde{b}} \left(D_C h_{CB} + \frac{1}{d-2} D_B h_{CC} \right) \right) \cong \\
& \cong -e\partial_\mu s_{\nu\sigma} \partial_\mu s_{\nu\sigma} + 2(1 - \tilde{a}^2) e\partial_\mu s_{\mu\sigma} \partial_\nu s_{\nu\sigma} - 2(1 + 2\tilde{a}\tilde{b}) e\partial_\mu s_{\mu\nu} \partial_\nu s_{\sigma\sigma} \\
& + (1 - 2\tilde{b}^2) e\partial_\mu s_{\nu\nu} \partial_\mu s_{\sigma\sigma} - eD_A s_{\mu\nu} D_A s_{\mu\nu} - \frac{\tilde{b}}{\tilde{a}} eD_A s_{\mu\mu} D_A s_{\nu\nu} \\
& - 2e\partial_\mu h_{\nu A} \partial_\mu h_{\nu A} - 2\frac{\tilde{a} + 2\tilde{b}}{\tilde{a}} e\partial_\mu h_{\mu A} \partial_\nu h_{\nu A} - 2eD_A h_{\mu B} D_A h_{\mu B} \\
& + 2eD_A h_{\mu B} D_B h_{\mu A} - \frac{2}{\tilde{a}^2} eD_A h_{\mu A} D_B h_{\mu B} - e\partial_\mu t_{AB} \partial_\mu t_{AB} - eD_C t_{AB} D_C t_{AB} \\
& + 2eR_{ABCD} t_{AC} t_{BD} - 2eR_{AB} t_{AC} t_{BC} - \frac{1}{2} eH_{ABCE} H_{ABDF} t_{CD} t_{EF} \\
& + \frac{\tilde{a} + 2\tilde{b}}{\tilde{a} + \tilde{b}} e \left(D_A t_{AC} + \left(\frac{1}{d-2} + \frac{1}{n} \right) D_C h_{AA} \right) \left(D_B t_{BC} + \left(\frac{1}{d-2} + \frac{1}{n} \right) D_C h_{BB} \right) \\
& - \left(\frac{1}{d-2} + \frac{1}{n} \right) e\partial_\mu h_{AA} \partial_\mu h_{BB} - \left(\frac{1}{d-2} + \frac{1}{n} \right) eD_C h_{AA} D_C h_{BB} \\
& - \frac{1}{2n^2} eH_{ABCD} H_{ABCD} h_{EE} h_{FF} + eh_{NN} [\bar{X}_8]_{1\Delta} + e[\bar{X}_8]_{2,\text{rd.}}
\end{aligned} \tag{64}$$

Here \tilde{a} and \tilde{b} are gauge parameters, (1) has been used to set $H_{ABCD} H_{ABCE} t_{DE}$ to 0, and $eD_A h_{\mu A} D_B h_{\mu B} - eR_{AB} h_{\mu A} h_{\mu B} \cong eD_A h_{\mu B} D_B h_{\mu A}$, valid for the metric (14), has

been used. If we choose $\tilde{a} = 1$, $\tilde{b} = -\frac{1}{2}$, we obtain de Donder gauge for $s_{\mu\nu}$ and Feynman gauge for $h_{\mu A}$, and the traceless tensor modes t_{AB} on \bar{H}^7 are decoupled from h_{AA} , apart from possible couplings coming from the last two terms.

The t_{AB} and h_{AA} modes are where tachyons are most likely to occur [116, 109]. If we ignore the last two terms in (64), then after making a Kaluza-Klein ansatz $t_{AB} = t(\tilde{x})\omega_{AB}(\bar{x})$, the intrinsic masses \bar{m} of the t_{AB} modes are determined by the spectrum on \bar{H}^7 in the metric $G_{AB} = B^2\bar{g}_{AB}$ of the equation:

$$\begin{aligned} & -D_C D_C \omega_{AB} - 2 R_{ACBD} \omega_{CD} + R_{AC} \omega_{CB} + R_{BC} \omega_{CA} \\ & + \frac{1}{2} H_{ACEF} H_{BDEF} \omega_{CD} - \frac{1}{2n} G_{AB} H_{CEFG} H_{DEFG} \omega_{CD} = \frac{\bar{m}^2}{B^2} \omega_{AB}. \end{aligned} \quad (65)$$

The last term in the left-hand side of (65) results from the tracelessness of t_{AB} in (64). From pages 42 to 43 of [20], the masses of these modes, as seen on the HW boundary, are $m = \frac{A}{B}\bar{m}$, where A and B are the constants in the metric ansatz (14). The first 4 terms in the left-hand side of (65) are known as the Lichnerowicz Laplacian acting on the traceless symmetric tensor ω_{AB} . On uncompactified H^n of sectional curvature $-\frac{1}{B^2}$ with $n > 2$, its spectrum extends from $\frac{(n-1)(n-9)}{4B^2}$ to $+\infty$ [117, 118]. If similar eigenfunctions with approximately the same eigenvalues exist on \bar{H}^n , then from (1), with the best estimate $\eta \simeq 1.425$ from the second paragraph before (23), where $\eta = \frac{h}{B^3}$ from (17), the spectrum of \bar{m}^2 in (65) on \bar{H}^7 would extend from about $-3 - 1.02 = -4.02$ to $+\infty$. Thus the t_{AB} modes would include tachyons, unless the last two terms in (64) lift their squared masses sufficiently.

The dilaton/radion is the mode of h_{AB} such that h_{AB} is an \bar{x} -independent multiple of G_{AB} , so that all covariant derivatives $D_A h_{BC}$ are 0, and $t_{AB} = 0$. For this mode $\bar{G}_{AB} = G_{AB} + 2h_{AB} = (B + \delta B)^2 \bar{g}_{AB}$, so $h_{AA} = 7 \left(\frac{\delta B}{B} + \frac{(\delta B)^2}{2B^2} \right)$. From just before (63), $\Lambda_{\text{SG}}^{(\text{bos})}$ is the integrand of $\Gamma_{\text{SG}}^{(\text{bos})}$ in (2), and after substituting for δB in terms of h_{AA} , the expansion of $\bar{\Lambda}_{\text{SG}}^{(\text{bos})}$ in powers of δB through quadratic order has the form:

$$\bar{\Lambda}_{\text{SG}}^{(\text{bos})} = \Lambda_{\text{SG}}^{(\text{bos})} + \frac{\partial \Lambda_{\text{SG}}^{(\text{bos})}}{\partial B} B \left(\frac{1}{7} h_{AA} - \frac{1}{98} h_{AA} h_{BB} \right) + \frac{1}{98} \frac{\partial^2 \Lambda_{\text{SG}}^{(\text{bos})}}{\partial B^2} B^2 h_{AA} h_{BB} \quad (66)$$

In the vacuum, $\Lambda_{\text{SG}}^{(\text{bos})}$ and $\frac{\partial \Lambda_{\text{SG}}^{(\text{bos})}}{\partial B}$ vanish by (18) and (19) respectively, and from (23), with the best estimate $\eta \simeq 1.425$ from the second paragraph before (23), $\frac{\partial^2 \Lambda_{\text{SG}}^{(\text{bos})}}{\partial B^2} \simeq -1.18 \frac{A^4 \sqrt{g}}{\kappa_{11}^{2/3} B^3}$. Thus after adding the momentum-dependent part of the dilaton/radion's kinetic term from $\frac{1}{2\kappa_{11}^2}$ times (64) with $d = 4$ and $n = 7$, neglecting any momentum-dependent contributions from the last two terms in (64), the terms quadratic in h_{AA}

in the gauge-fixed $\bar{\Lambda}_{\text{SG,g.f.}}^{(\text{bos})}$, in a gauge with $\tilde{a} + 2\tilde{b} = 0$, are:

$$\bar{\Lambda}_{\text{SG,g.f.}}^{(\text{bos})} \simeq -\frac{A^4 B^7 \sqrt{g}}{2\kappa_{11}^2} \left(\frac{9}{14} \partial_\mu h_{AA} \partial_\mu h_{BB} + \frac{1}{49} \frac{1.18 \kappa_{11}^{4/3}}{B^8} h_{AA} h_{BB} \right). \quad (67)$$

Thus using the best estimate $B \simeq 0.28 \kappa_{11}^{2/9}$ from the second paragraph before (23), the intrinsic mass squared of the dilaton/radion is $\simeq 78$, so from pages 42 to 43 of [20], the dilaton/radion's mass, as seen on the HW boundary, is $m_{\text{dil}} \simeq 9 \frac{A}{B}$.

The dilaton/radion's intrinsic mass squared receives a contribution $\simeq 1.4$ from the third from last term in (64), and the remaining $\simeq 76.3$ comes from the last term in (64), so its relatively large size suggests that the last two terms in (64) might be able to raise the squared intrinsic masses of the t_{AB} modes sufficiently to avoid the occurrence of tachyons. The relatively large value of the mass term in (67) is due to the relatively small value of $\frac{B}{\kappa_{11}^{2/9}} \simeq 0.28$, notwithstanding that this value of $\frac{B}{\kappa_{11}^{2/9}}$ satisfies the Giudice-Rattazzi-Wells perturbativity criterion [57] by a substantial margin, as noted in the paragraph before (23). It is interesting to note that the above estimate of the dilaton/radion mass is about 42 times larger than the mass $\simeq 0.2 \frac{A}{B}$ of the classically massless harmonic 3-form modes found in subsection 2.1.1.

If we ignore the last two terms in (64), then after making a Kaluza-Klein ansatz $h_{\mu A} = h_\mu(\tilde{x}) \omega_A(\bar{x})$, the intrinsic masses \bar{m} of the $h_{\mu A}$ modes are determined by the spectrum on \bar{H}^7 in the metric \bar{g}_{AB} of sectional curvature -1 of the equation $-(\delta d + d\delta)\omega = \bar{m}^2 \omega$, where $\delta d + d\delta$ is the Hodge - de Rham Laplacian for 1-forms on \bar{H}^7 . So from pages 9 to 12 and 16 to 17 of [20], it seems likely that the lightest massive modes of $h_{\mu A}$ will have intrinsic mass $\bar{m} = 2$, up to corrections from the last two terms in (64) [12, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 16, 92, 93]. And from pages 42 to 43 of [20], their masses in this approximation, as seen on the HW boundary, are $m = 2 \frac{A}{B}$, where A and B are the constants in the metric ansatz (14). However the relatively large contribution of the last term in (64) to the dilaton/radion's mass squared suggests that the last two terms in (64) might give a larger contribution to \bar{m}^2 for these modes than the value 4 obtained in this first approximation.

There are no massless vector modes $h_{\mu A}$ corresponding to continuous symmetries of \bar{H}^7 , because \bar{H}^7 is a smooth compact negatively curved Einstein space, and therefore cannot have any continuous symmetries. For a vector field V^A that generates a continuous symmetry on a smooth Riemannian manifold \mathcal{M} satisfies the Killing vector equation $D_A V_B + D_B V_A = 0$, and thus $0 = D^A (D_A V_B + D_B V_A)$. But from the

definition (5) of the Riemann tensor, on page 4, we have $D^A D_B V_A = D_B D^A V_A + R_{BD} V^D$, and from the Killing vector equation, we have $D^A V_A = 0$. And if \mathcal{M} is a negatively curved n -dimensional Einstein space with $n > 2$, then $R_{BD} = -\alpha g_{BD}$, where $\alpha = -\frac{1}{n}R > 0$ is independent of position by the contracted Bianchi identity, $D_A R - 2D^B R_{AB} = -(n-2)\partial_A \alpha = 0$. Thus we find $D^A D_A V_B = \alpha g_{BD} V^D$, hence $V^B D^A D_A V_B = \alpha V^B g_{BD} V^D$. Thus if \mathcal{M} is compact, we find on integrating by parts that:

$$\int_{\mathcal{M}} d^d x (D^A V^B) (D_A V_B) = -\alpha \int_{\mathcal{M}} d^d x V^B g_{BD} V^D \quad (68)$$

The left-hand side of this equation is ≥ 0 , but for nonzero V^A , the right-hand side is < 0 , so there can be no such nonzero V^A .

The classically massless $h_{\mu A}$ modes corresponding to harmonic 1-forms on \bar{H}^7 could obtain masses from terms in the last term in (64) built from $h_{\mu A} h_{\mu B}$ and the vacuum R_{ABCD} and H_{ABCD} , as well as from further quantum corrections, like the harmonic p -form modes of C_{IJK} .

If we ignore the last two terms in (64), then after making a Kaluza-Klein ansatz $s_{\mu\nu} = s_{\mu\nu}(\tilde{x}) \omega(\tilde{x})$, the intrinsic masses \bar{m} of the $s_{\mu\nu}$ modes are determined by the spectrum on \bar{H}^7 in the metric \bar{g}_{AB} of sectional curvature -1 of the negative of the Laplace-Beltrami operator: $-\frac{1}{\sqrt{\bar{g}}} \partial_A (\sqrt{\bar{g}} g^{AB} \partial_B \omega) = \bar{m}^2 \omega$. So from pages 9 to 12 and 16 to 17 of [20], it seems likely that in this approximation, the lightest massive modes of $s_{\mu\nu}$ will have intrinsic mass $\bar{m} = 3$ [12, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 16, 92, 93], and from pages 42 to 43 of [20], their masses as seen on the HW boundary will be $m = 3\frac{A}{B}$, where A and B are the constants in the metric ansatz (14). But as for the $h_{\mu A}$ modes, the relatively large contribution of the last term in (64) to the dilaton/radion's mass squared suggests that the last two terms in (64) might give a larger contribution to \bar{m}^2 for these modes than the value 9 obtained in this first approximation.

If we make the limiting gauge choice $\tilde{b} \rightarrow -\tilde{a} + \tilde{a}^2$, $\tilde{a} \rightarrow 0$ in (64), we obtain the Fierz-Pauli unitary gauge for the massive $s_{\mu\nu}$ modes and the Proca unitary gauge for the massive $h_{\mu A}$ modes along the extended dimensions [119, 95, 96], and Landau-gauge-like restrictions $D_A h_{\mu A} = 0$ and $D_A t_{AC} + \left(\frac{1}{d-2} + \frac{1}{n}\right) D_C h_{AA} = 0$ on the dependence of the modes on position on \bar{H}^7 , which in the same way as for the 3-form gauge field means that some of the massive modes obtained in the de Donder/Feynman gauge are unphysical, and would be cancelled by corresponding Faddeev-Popov ghosts in the de Donder/Feynman gauge. The gauge invariance for the massless $s_{\mu\nu}$ modes is unfixed

in the limiting unitary gauge, and we are free to add additional gauge-fixing terms just for these modes, which are independent of position on \bar{H}^7 .

3 Modes that decay along the beam line outside the interaction region at the LHC

The classically massless harmonic 3-form modes were found in subsection 2.1.1 above to acquire approximately equal masses $\simeq 0.2\frac{A}{B}$, and in subsection 2.1.2 their leading couplings (56) to the SM gauge bosons were found to be axion-like near their mass shell. From pages 17 to 19 of [20], their number is expected to be $\sim \frac{\bar{V}_7}{\ln \bar{V}_7}$, so they could be sufficiently numerous for their large number to compensate for the $\sim \frac{1}{\sqrt{\bar{V}_7}}$ suppression of their couplings enough for them to be seen at the LHC.

From pages 12 to 13 of [20], it seems possible, on the basis of the results of [120] and [121], that the lightest generic classically massive modes of $C_{\mu\nu\sigma}$, $C_{\mu\nu A}$, $C_{\mu AB}$, $s_{\mu\nu}$, h_{AA} , and $h_{\mu A}$, with classical masses $3\frac{A}{B}$, $2\frac{A}{B}$, $\frac{A}{B}$, $3\frac{A}{B}$, $3\frac{A}{B}$, and $2\frac{A}{B}$ respectively, could have large degeneracies $\sim \bar{V}_7$, that restore agreement between the spectral staircase and the Weyl asymptotic formula for the number of modes up to mass m , immediately above the generic spectral gap. Any non-generic lighter modes would be too few to see at the LHC. However the much larger mass $\simeq 9\frac{A}{B}$ calculated in subsection 2.2 above for the dilaton/radion mode of h_{AA} , which is classically massless, suggests that when the contributions of the $\Gamma_{\text{SG}}^{(8,\text{bos})}$ term (4) in (2) are included, the lightest classically massive modes of all the above types, and also the lightest modes of t_{AB} and the lightest modes of C_{ABC} other than the harmonic 3-form modes, might all have masses $\sim 8\frac{A}{B}$ or more.

The harmonic 1-form, 2-form, and 3-form modes of $C_{\mu\nu A}$, $C_{\mu AB}$, and C_{ABC} respectively have vanishing field strength H_{IJKL} , and thus cannot get masses directly from the CJS action (3) or $\Gamma_{\text{SG}}^{(8,\text{bos})}$. The leading contributions to the masses of these modes come from loop diagrams that contain two CJS Chern-Simons vertices, whose contribution to the mass of the harmonic 3-form modes of C_{ABC} was approximately calculated in subsection 2.1.1 above. It is the fact that C_{ABC} has no classical spectral gap [85, 91] that results in the number of harmonic 3-form modes being $\sim \frac{\bar{V}_7}{\ln \bar{V}_7}$, while the numbers of harmonic 2-form and 1-form modes are $\sim \bar{V}_7^\alpha$, with $\alpha < 1$ [99, 100].

Thus it seems possible that the model considered here and in [20] has several types of approximately degenerate bosonic modes of the supergravity multiplet, such that

the numbers of approximately degenerate modes of each type are large enough to compensate for the $\sim \frac{1}{\sqrt{V_7}}$ suppression of their couplings to the SM states, so that the modes could be seen at the LHC. The lightest such modes are likely to be the classically massless harmonic 3-form modes of C_{ABC} , with mass $\simeq 0.2 \frac{A}{B}$, and axion-like couplings to the SM gauge bosons.

3.1 Approximate distribution of decay lengths

Considering now the modes of just one of these types, approximately degenerate with mass m , the modes will not be exactly degenerate, because the root mean square field strength h of the vacuum 4-form fluxes is likely to vary slightly from region to region on \bar{H}^7 , and moreover for the harmonic 3-form modes, the degeneracy depends on the approximation discussed in the second paragraph after (44). The variation of h from region to region on \bar{H}^7 will be random, so by Anderson localization [122], the modes will be approximately localized on different regions of \bar{H}^7 . For the following rough estimates I shall treat the modes as if they were scalars, both along the 4 extended dimensions and along \bar{H}^7 . The spherically symmetric eigenmodes of the Laplace-Beltrami operator on uncompactified H^7 behave like $e^{-3\bar{y}}$ times an oscillating factor at large intrinsic geodesic distance \bar{y} from their centre of spherical symmetry [123, 124], so I shall assume that the approximately localized modes behave roughly as $e^{-\left(3+\frac{1}{L_{\text{loc}}}\right)\bar{y}}$, where \bar{y} is the intrinsic geodesic distance from their centre of localization, and the intrinsic localization length \bar{L}_{loc} is likely to be $\sim \frac{\bar{L}_7}{2}$ for the harmonic 3-form modes, which from the discussion after (41) above is ~ 14 if \bar{H}^7 is reasonably isotropic, and M_{11} is near its current lower limit of 2.3 ± 0.7 TeV for 7 flat extra dimensions.

The coupling constant c of one of these modes to the SM fields is roughly the amplitude $e^{-\left(3+\frac{1}{L_{\text{loc}}}\right)\bar{y}}$ of the mode at the HW boundary times a constant that is the same for all the modes of this type, as for example in (56) above, where \bar{y} is now the intrinsic geodesic distance from the HW boundary to the localization centre of the mode. The s channel production rate of such a mode is proportional to c^2 , and its width is also proportional to c^2 , so its lifetime is proportional to $\frac{1}{c^2}$. If such a mode is produced at the interaction point (IP) in ATLAS or CMS, it will have a longitudinal momentum along the beam direction equal to the net longitudinal momentum of the two partons that produced it. The distribution of longitudinal momentum of the mode, and of the corresponding relativistic enhancement factor for its lifetime in the

laboratory frame, are independent of c , so for a rough estimate I shall treat the mode as having a fixed decay length l along the beam direction, that is $\frac{1}{c^2}$ times a constant that is the same for all the modes of this type.

For reasonably isotropic \bar{H}^7 , the intrinsic volume of \bar{H}^7 between intrinsic geodesic distances \bar{y} and $\bar{y} + d\bar{y}$ from a fixed point of \bar{H}^7 is from page 9 of [20] roughly a constant times $e^{6\bar{y}} d\bar{y}$, for \bar{y} up to its maximum value $\simeq \frac{\bar{L}_7}{2}$. Thus the number of modes whose coupling constant to the SM fields is between c and $c + dc$ is roughly $c^{-\frac{6\bar{L}_{\text{loc}}}{3\bar{L}_{\text{loc}}+1}} \frac{dc}{c}$, times a constant that is the same for all the modes of this type. So the number of modes whose decay length along the beam direction is between l and $l + dl$ is roughly $l^{\frac{3\bar{L}_{\text{loc}}}{3\bar{L}_{\text{loc}}+1}} \frac{dl}{l}$, times a constant that is the same for all the modes of this type.

The number of particles of decay length l decaying between distances z and $z + dz$ along the beam line from the IP per unit time, summed over both directions along the beam, is $\frac{dz}{l} e^{-z/l}$ times the production rate of particles of decay length l . The production rate of one of the approximately degenerate modes of mass m at the LHC is proportional to c^2 , thus is $\frac{1}{l}$ times a constant that is the same for all the modes of this type. Thus the number of particles of this type decaying between distances z and $z + dz$ along the beam line from the IP per unit time is:

$$k_1 dz \int_{l_{\min}}^{l_{\max}} \frac{1}{l} e^{-z/l} \frac{1}{l} l^{\frac{3\bar{L}_{\text{loc}}}{3\bar{L}_{\text{loc}}+1}} \frac{dl}{l} \simeq k_2 z^{-\frac{3\bar{L}_{\text{loc}}+2}{3\bar{L}_{\text{loc}}+1}} \left(1 - e^{-z/l_{\min}}\right) dz, \quad (69)$$

where k_1 and k_2 are constants that are the same for all modes of this type, $l_{\min} \sim 10^{-19}$ metres, l_{\max} is determined by the maximum value $\simeq \frac{\bar{L}_7}{2}$ of \bar{y} , and in the right hand side of (69) I have set l_{\max} to ∞ since the integral of (69) over z is convergent at large z , and used an approximation for the incomplete Γ function.

The integral of the right-hand side of (69) over z from 0 to ∞ is convergent at both limits, and by integration by parts is equal to:

$$k_2 \left(3\bar{L}_{\text{loc}} + 1\right) \Gamma\left(\frac{3\bar{L}_{\text{loc}}}{3\bar{L}_{\text{loc}} + 1}\right) l_{\min}^{-\frac{1}{3\bar{L}_{\text{loc}}+1}} \simeq k_2 \left(3\bar{L}_{\text{loc}} + 1\right) l_{\min}^{-\frac{1}{3\bar{L}_{\text{loc}}+1}}, \quad (70)$$

where the right hand side of (70) is approximately valid for $\bar{L}_{\text{loc}} > 1$.

Thus even though the vast majority of the approximately degenerate modes of mass $\simeq m$ couple only with gravitational strength to the SM fields, so that their lifetimes are $\sim \frac{1}{G_N m^3} \sim \text{hours}$ [57], the approximate localization of the modes on \bar{H}^7 , and the inverse proportionality of the production rate of the modes to their decay length, mean that most of the modes actually produced at the LHC decay in or near the detectors,

so that the Breit-Wigner formula [107, 125] can be used to calculate the total cross-section for the production and decay of these modes in the s channel, as implicitly assumed on pages 13 to 16 of [20]. The possibility that modes seen at the LHC could be a relatively small number of linear combinations, localized near the HW boundary, of the large number of approximately degenerate modes, and that the modes localized near the HW boundary would have correspondingly large couplings to the SM fields, was noted on page 43 of [20].

3.2 Approximate total cross-section

For comparison with the LHC data, let N be the number of approximately degenerate modes of mass $\simeq m$, and μ be the width of the distribution over which the modes are spread. The N approximately degenerate modes of mass $\simeq m$ will be labelled by an index n . I will assume that $\mu \geq \Gamma_n$ for all the modes $|n\rangle$, where Γ_n is the total width of the mode $|n\rangle$.

A monoenergetic high energy beam of protons of energy E is equivalent to a beam of partons, such that the number of u quarks per unit area per unit time with energy between xE and $(x + dx)E$ is $f_u(x) dx$ times the number of protons per unit area per unit time, and similarly for the other types of parton, where $f_p(x)$, $p = u, d, g, \bar{u}, \bar{d}, \dots$ are the parton distribution functions (PDFs). The PDFs evolve logarithmically with Q^2 , the square of the momentum transferred in a scattering process, and for a rough estimate I shall use the plot [126] with $Q^2 = 10^4 \text{ GeV}^2$.

The plot [126] shows that to a good approximation, $f_g(x) \geq 10f_{\bar{u}}(x)$ for all x , $f_g(x) \geq f_u(x)$ for all $x \leq 0.1$, $f_g(x) \geq 0.1f_u(x)$ for x up to at least 0.4, $f_g(x) \geq 5f_{\bar{d}}(x)$ for all x , $f_g(x) \geq f_d(x)$ for $x \leq 0.25$, and $f_g(x) \geq 0.2f_d(x)$ for x up to at least 0.35. The modes of mass $\simeq m$ are uncharged, and couple with approximately equal strength to all the partons, so for a first approximation, valid for m up to at least about 2.5 TeV for the LHC with 7 TeV centre of mass energy, I shall consider the gg initial state only.

If N was 1 and all the modes of mass $\simeq m$ produced in the s channel decayed within the detector, then the total cross-section for the process $gg \rightarrow n \rightarrow f$, where $|n\rangle$ is the mode of mass $\simeq m$, which for the rough estimates here I am treating as if it was a scalar, and f is an SM final state such as $u\bar{u}$, gg , W^+W^- , \dots , would be given

in the $|n\rangle$ rest frame by the Breit-Wigner formula [107, 125]:

$$\sigma(gg \rightarrow n \rightarrow f) = \frac{1}{8} \cdot \frac{1}{2^2} \cdot \frac{1}{2} \cdot 4\pi \cdot \frac{\Gamma_{n \rightarrow gg} \Gamma_{n \rightarrow f}}{E^2} \cdot \frac{1}{((E - m_n)^2 + \Gamma_n^2/4)}, \quad (71)$$

where E is the invariant mass of the gg system, and $\Gamma_{n \rightarrow gg}$ and $\Gamma_{n \rightarrow f}$ are the partial widths for $|n\rangle$ decay to gg and f . The factor $\frac{1}{8}$ is the probability that the two initial gluons can form a colour singlet, the factor $\frac{1}{2^2}$ is for the average over the helicity states of the initial gluons, and the factor $\frac{1}{2}$ is the probability that the helicities of the two initial gluons sum to 0.

For each type of SM final state f , $\Gamma_{n \rightarrow f} = \int |\langle f | n \rangle|^2 d\rho_f$, where $\int \dots d\rho_f$ represents a phase space integration that is independent of n [107, 125]. The mode $|n\rangle$ is not present in the final state, so for $N > 1$ we have to sum the amplitude over all the modes $|n\rangle$. The amplitude factor that leads to the final two factors in $\sigma(gg \rightarrow n \rightarrow f)$, (71), after the phase space integrations are done, is:

$$\frac{1}{E} \langle f | n \rangle \langle n | gg \rangle \frac{1}{E - m_n + i\Gamma_n/2}. \quad (72)$$

In the example (56), the only dependence on the mode $|n\rangle$ of the matrix element $\langle f | n \rangle$, for SM final states f consisting of SM gauge bosons, whose wave functions are independent of position on the closed hyperbolic factor \bar{H}^6 of the HW boundary, is effectively via a single coupling constant that measures the integral of $|n\rangle$ over the HW boundary, weighted by the vacuum 4-form fluxes at the boundary. If the final state f includes quarks or leptons, whose wave functions depend on position on \bar{H}^6 , different $|n\rangle$, whose wave functions are larger or smaller in different regions of \bar{H}^6 , could couple with different effective coupling constants to different quarks and leptons. But from the discussion after (41), on page 14, the intrinsic diameter \bar{L}_6 of \bar{H}^6 lies between about 5.7 and 6.0 if \bar{H}^6 is reasonably isotropic, and the current upper bound on the intrinsic diameter \bar{L}_7 of \bar{H}^7 is about 28 if \bar{H}^7 is reasonably isotropic, so since the curvature radius b_1 of \bar{H}^6 is $\simeq B$, the ratios \bar{L}_7/\bar{L}_6 and L_7/L_6 lie between about 4.9 and 4.7, if both \bar{H}^6 and \bar{H}^7 are reasonably isotropic, and \bar{L}_7 is near its current upper bound.

I shall therefore assume that for a first approximation, the only dependence of the matrix element $\langle f | n \rangle$ on the mode $|n\rangle$ is via a single real-valued effective coupling constant $c \geq 0$ that measures how large the wave function of $|n\rangle$ is at the HW boundary. We now choose a particular element of the eigenmode basis that is localized close to \bar{H}^6 , say $|1\rangle$. For each mode $|n\rangle$, we define the coupling constant c_n of $|n\rangle$ to the SM

states to be such that for one particular SM final state $\langle f_0|$, $\langle f_0|n\rangle = \langle f_0|1\rangle c_n$. Then for all SM final states $\langle f|$:

$$\langle f|n\rangle \simeq \langle f|1\rangle c_n. \quad (73)$$

Let $\hat{\rho}_2(c, m')$ be such that the number of elements $|n\rangle$ of the eigenmode basis such that c_n lies between c and $c + dc$, and the exact mass m_n of $|n\rangle$ lies between m' and $m' + dm'$, is $\hat{\rho}_2(c, m') dcdm'$. I shall assume that due to the random and uncorrelated nature of the slight variations from region to region on \bar{H}^7 of the root mean square field strength h of the vacuum 4-form fluxes, $\hat{\rho}_2(c, m')$ approximately factorizes as:

$$\hat{\rho}_2(c, m') \simeq \hat{\rho}_c(c) \hat{\rho}_m(m'), \quad (74)$$

where

$$\int_0^\infty dc \hat{\rho}_c(c) = 1, \quad \int_{m-\mu/2}^{m+\mu/2} \hat{\rho}_m(m') dm' = N. \quad (75)$$

Then:

$$\begin{aligned} \sum_n \langle f|n\rangle \langle n|gg\rangle &\simeq \langle f|1\rangle \langle 1|gg\rangle \int_0^\infty dc \int_{m-\mu/2}^{m+\mu/2} dm' c^2 \hat{\rho}_2(c, m') \simeq \\ &\simeq N \langle f|1\rangle \langle 1|gg\rangle \int_0^\infty dc c^2 \hat{\rho}_c(c). \end{aligned} \quad (76)$$

The number N of approximately degenerate modes $|n\rangle$ of mass $\simeq m$ would be at most $\sim \bar{V}_7$, which for $m \sim \kappa_{11}^{-2/9} \sim \text{TeV}$ is $\sim 10^{32}$. I shall assume that $\hat{\rho}_m(m')$ is smooth, and is 0 for $|m' - m| \geq \mu/2$.

The only factor in the amplitude factor (72) that varies significantly with m_n over the mass range $m - \mu/2 \leq m_n \leq m + \mu/2$ is the final factor:

$$\frac{1}{E - m_n + i\Gamma_n/2}, \quad (77)$$

where by assumption $\Gamma_n \leq \mu$. The sum of the amplitude factor (72) over the modes $|n\rangle$ can be replaced by integrals over $c = c_n$ and $m' = m_n$ as in (76). To a first approximation, when multiplied by the smooth density of states $\hat{\rho}_2(c, m') \simeq \hat{\rho}_c(c) \hat{\rho}_m(m')$ and integrated over m' , the m' -dependent factor (77) is effectively $\simeq -i\pi\delta(E - m')$, because the $i\Gamma_n/2$ means that the integration path has to go around the singularity in the lower half of the complex m' plane. Choosing the integration path to be along the real axis except for a small semicircle centred at $m' = E$, the contributions from the real axis cancel to a good approximation for $\Gamma_n \ll \mu$, and roughly cancel for all $\Gamma_n \leq \mu$, and the semicircle gives $-i\pi$ times the density of states $\hat{\rho}_c(c) \hat{\rho}_m(m')$ evaluated at $m' = E$. Thus the sum of the amplitude factor (72) over the modes $|n\rangle$ is

approximately:

$$\begin{aligned}
& \sum_n \frac{1}{E} \langle f | n \rangle \langle n | gg \rangle \frac{1}{E - m_n + i\Gamma_n/2} \simeq \\
& \simeq -i\pi \frac{1}{E} \langle f | 1 \rangle \langle 1 | gg \rangle \int_0^\infty dc c^2 \hat{\rho}_c(c) \int_{m-\mu/2}^{m+\mu/2} dm' \hat{\rho}_m(m') \delta(E - m') \simeq \\
& \simeq -i\pi \frac{1}{E} \hat{\rho}_m(E) \langle f | 1 \rangle \langle 1 | gg \rangle \int_0^\infty dc c^2 \hat{\rho}_c(c) \simeq \\
& \simeq -i\pi \frac{1}{E} \hat{\rho}_m(E) \frac{1}{N} \sum_n \langle f | n \rangle \langle n | gg \rangle. \tag{78}
\end{aligned}$$

where (76) was used at the last step.

For N modes, the final two factors in (71) are replaced by the phase space integrals $\int \int \dots d\rho_f d\rho_{gg}$ of the squared magnitude of the mode sum (78) of the amplitude factor (72), which are approximately:

$$\begin{aligned}
& \pi^2 \frac{1}{E^2} \hat{\rho}_m(E)^2 \frac{1}{N^2} \int \int \sum_n \langle f | n \rangle \langle n | gg \rangle \sum_{n'} \langle gg | n' \rangle \langle n' | f \rangle d\rho_f d\rho_{gg} \simeq \\
& \simeq \pi^2 \frac{1}{E^2} \hat{\rho}_m(E)^2 \frac{1}{N^2} \int \int \sum_n \sum_{n'} c_n^2 c_{n'}^2 \langle f | 1 \rangle \langle 1 | gg \rangle \langle gg | 1 \rangle \langle 1 | f \rangle d\rho_f d\rho_{gg} \simeq \\
& \simeq \pi^2 \frac{1}{E^2} \hat{\rho}_m(E)^2 \frac{1}{N^2} \int \sum_n \langle f | n \rangle \langle n | f \rangle d\rho_f \int \sum_{n'} \langle gg | n' \rangle \langle n' | gg \rangle d\rho_{gg}, \tag{79}
\end{aligned}$$

where the approximate factorization (73) has been used at each step.

In addition to the eigenmode basis of the modes $|n\rangle$ of mass $\simeq m$, we can consider a basis where all the modes are approximately uniformly spread out over \bar{H}^7 , and thus by (41) couple with approximately equal, gravitational, strength to the SM fields. Let $\{|\bar{n}\rangle\}$ be a basis of this type. It is related to the eigenmode basis by an $N \times N$ unitary transformation.

The partial widths $\Gamma_{\bar{n} \rightarrow f} \equiv \int |\langle f | \bar{n} \rangle|^2 d\rho_f$ of the modes in the $\{|\bar{n}\rangle\}$ basis are estimated in order of magnitude by [57]:

$$\Gamma_{\bar{n} \rightarrow gg} \sim \Gamma_{\bar{n} \rightarrow ggg} \sim \dots \sim \Gamma_{\bar{n} \rightarrow u\bar{u}} \sim \dots \sim \Gamma_{\bar{n} \rightarrow \nu_\tau \bar{\nu}_\tau} \sim m^3 G_N \sim 10^{-32} \text{TeV}, \tag{80}$$

where the final \sim applies for $m \sim \text{TeV}$. Thus for all SM final states f :

$$\int \sum_n \langle f | n \rangle \langle n | f \rangle d\rho_f = \int \sum_{\bar{n}} \langle f | \bar{n} \rangle \langle \bar{n} | f \rangle d\rho_f \sim N m^3 G_N. \tag{81}$$

Thus from (79), the last two factors in (71) are for the N modes replaced by roughly:

$$\sim \pi^2 \frac{1}{E^2} \hat{\rho}_m(E)^2 m^6 G_N^2. \tag{82}$$

Thus the total cross-section is roughly:

$$\sigma(gg \rightarrow \text{any } n \rightarrow f) = \frac{1}{8} \cdot \frac{1}{2^2} \cdot \frac{1}{2} \cdot 4\pi \cdot \pi^2 \frac{1}{E^2} \hat{\rho}_m(E)^2 m^6 G_N^2 \sim m^4 G_N^2 \hat{\rho}_m(E)^2. \quad (83)$$

To produce a mode $|n\rangle$ with mass $\simeq m$ at rest with $P = 3.5$ TeV per proton, each gluon needs $x \simeq \frac{m}{2P}$. If the 4-momenta of the protons are $(P, 0, 0, P)$ and $(P, 0, 0, -P)$ and the momentum fractions of the gluons are x_1 and x_2 , then their Mandelstam s is $P^2((x_1 + x_2)^2 - (x_1 - x_2)^2) = 4P^2 x_1 x_2$. From the plot [126] we find that $f_g(x) \simeq 0.060x^{-2.17}$ for $0.05 \leq x \leq 0.2$, but substantially smaller than this for $x \geq 0.3$. For a first approximation I shall use $f_g(x) \simeq 0.060x^{-2.17}$ for $0 \leq x \leq 0.3$ and $f_g(x) \simeq 0$ for $0.3 < x \leq 1$. The initial partons are massless, so the final form of the estimate (83) of the total cross-section in the centre of mass frame of the two gluons is also the approximate total cross-section in the laboratory frame [125]. Thus the total cross-section for proton + proton \rightarrow any $n + X \rightarrow f + X$ is roughly:

$$\begin{aligned} & \int_0^{0.3} dx_1 \int_0^{0.3} dx_2 0.060^2 (x_1 x_2)^{-2.17} m^4 G_N^2 \hat{\rho}_m(2P\sqrt{x_1 x_2})^2 \\ & \simeq 0.060^2 m^4 G_N^2 \frac{N^2}{\mu^2} \int_{\frac{m^2}{1.2P^2}}^{0.3} \frac{dx_1}{x_1} \int_{\frac{2m-\mu}{4P}}^{\frac{2m+\mu}{4P}} \frac{2x dx}{x^{4.34}} \\ & \simeq 0.1 m^3 G_N^2 \frac{N^2}{\mu} \left(\frac{P}{m}\right)^{2.34} \ln \frac{0.6P}{m}, \end{aligned} \quad (84)$$

where I approximated $\hat{\rho}_m(E)$ as $\frac{N}{\mu}$ from $m - \frac{\mu}{2}$ to $m + \frac{\mu}{2}$ and 0 outside this interval, defined $x \equiv \sqrt{x_1 x_2}$, and assumed $\mu \ll m$ and $\frac{m}{2P} \leq 0.3$.

As a reference estimate of the number N of approximately degenerate modes of mass $\simeq m$, let $N_{\text{Weyl}, \bar{m}}$ be the Weyl asymptotic formula for the number of eigenmodes of the negative of the Laplace-Beltrami operator $\Delta = \frac{1}{\sqrt{\bar{g}}} \partial_A (\sqrt{\bar{g}} \bar{g}^{AB} \partial_B \cdot)$ on \bar{H}^7 , in the metric \bar{g}_{AB} of sectional curvature -1 , with eigenvalue up to \bar{m}^2 , where $\bar{m} = m \frac{B}{A}$ is the intrinsic mass corresponding to m :

$$N_{\text{Weyl}, \bar{m}} = \frac{\bar{V}_7}{(2\pi)^7} S_6 \frac{\bar{m}^7}{7} = \frac{\bar{V}_7}{840\pi^4} \bar{m}^7, \quad (85)$$

where $S_6 = \frac{16}{15}\pi^3$ is the area of the unit 6-sphere. Thus from (41):

$$N_{\text{Weyl}, \bar{m}} G_N = \frac{\kappa_{11}^2 \bar{m}^7}{6720\pi^5 A^2 B^7} \simeq 0.046 \frac{\bar{m}^9}{m^2}, \quad (86)$$

where the best value $B \simeq 0.28\kappa_{11}^{2/9}$, from subsection 1.1, has been used.

In numerical studies of the spectrum of $-\Delta$ on compact hyperbolic 3-manifolds of small intrinsic volume, Inoue found that the Weyl asymptotic formula is approximately valid down to the lowest non-zero eigenvalue λ_1 , so that if λ_1 occurs at a larger value of \bar{m} than would be expected from the Weyl formula, there is a degeneracy or approximate degeneracy of eigenvalues near λ_1 , that restores agreement with the Weyl formula for \bar{m}^2 above λ_1 [120]. It is also known that for $n \geq 2$, every \bar{H}^n has pairs of finite covers of arbitrarily large volume ratio, whose sets of eigenvalues of $-\Delta$, ignoring multiplicities, are identical [121], so since the Weyl asymptotic formula is certainly valid for sufficiently large \bar{m} , every \bar{H}^n has finite covers whose eigenvalues have arbitrarily large multiplicities, for sufficiently large \bar{m} .

It seems likely that $N_{\text{Weyl},\bar{m}}$, (85), will give an under-estimate of N for the classically massless harmonic 3-form modes, whose intrinsic mass was calculated approximately as $\bar{m} \simeq 0.2$ in subsection 2.1.1, and an over-estimate for all the other types of mode. For the harmonic 3-form modes, N is the 3rd Betti number B_3 of \bar{H}^7 , which from pages 17 to 19 of [20], is expected to be $\sim \frac{\bar{V}_7}{\ln \bar{V}_7}$. For a reference estimate of the coefficient, the middle Betti number of an \bar{H}^n with even n and large intrinsic volume \bar{V}_n is from pages 18 to 19 of [20] given roughly by $B_{n/2} \simeq \frac{(n-1)!!}{(2\pi)^{n/2}} \bar{V}_n$, and in particular, for $n = 6$, $B_3 \simeq 0.060 \bar{V}_6$, and for $n = 8$, $B_4 \simeq 0.067 \bar{V}_8$ [99, 100]. I shall use $N_{3\text{-form,ref}} \equiv 0.06 \frac{\bar{V}_7}{\ln \bar{V}_7}$ as a reference estimate of B_3 of \bar{H}^7 , which for $\bar{V}^7 \sim 10^{34}$, gives $N_{3\text{-form,ref}} \simeq 8 \times 10^{-4} \bar{V}_7$, while from (85), $N_{\text{Weyl},0.2} \simeq 2 \times 10^{-10} \bar{V}_7$.

The numbers of harmonic 2-form and 1-form modes are $\sim \bar{V}_7^\alpha$, with $\alpha < 1$ [99, 100], so these modes are not expected to be observable at the LHC.

For the remaining modes, the much larger intrinsic mass $\bar{m}_{\text{dil}} \simeq 9$ calculated in subsection 2.2 for the dilaton/radion mode of h_{AA} , which is classically massless, suggests that the $\Gamma_{\text{SG}}^{(8,\text{bos})}$ term (4) in (2) might give an additive contribution $\sim 9^2$ to the squares of their intrinsic masses, so that a better estimate of N might be obtained by replacing \bar{m} in (85) by $\sqrt{\bar{m}^2 - \bar{m}_0^2}$, for some $\bar{m}_0 < \bar{m}$.

Substituting $N_{\text{Weyl},\bar{m}}$ for N in (84) and using (86), we obtain:

$$\sigma(\text{prot} + \text{prot} \rightarrow \text{any } n + X \rightarrow f + X) \sim 10^{-4} \frac{\bar{m}^{18}}{\mu m} \left(\frac{P}{m}\right)^{2.34} \ln \frac{0.6P}{m}, \quad (87)$$

as a rough reference estimate of the total cross-section for the modes of each type except the harmonic 3-forms, for $m < 0.6P$, where P is the energy per proton, currently 3.5 TeV at the LHC. For the harmonic 3-forms, the reference estimate of N is $N_{3\text{-form,ref}}$ not $N_{\text{Weyl},0.2}$, so the coefficient 10^{-4} in the right-hand side of (87) is replaced by 10^9 .

3.3 LHC results and prospects

From (69) and (70), the fraction of the events where one of the N approximately degenerate modes $|n\rangle$ of mass $\simeq m$ is produced in the s channel, such that the $|n\rangle$ decays further than $z_{\text{cut}} \gg l_{\text{min}}$ along the beam line from the IP, is $\simeq \left(\frac{z_{\text{cut}}}{l_{\text{min}}}\right)^{-\frac{1}{3L_{\text{loc}}+1}}$. Relevant searches at ATLAS and CMS have so far always accepted events that pass all cuts not related to the position z of the reconstructed primary vertex along the beam line relative to the IP, and for which $|z| \leq 10$ centimetres [127, 128, 69, 129, 130]. Thus if the intrinsic localization length \bar{L}_{loc} was 28, which from the start of subsection 3.1, is twice the largest expected value $\simeq \frac{\bar{L}_T}{2} \simeq 14$ for M_{11} at its current lower limit for 7 flat extra dimensions, the fraction of $|n\rangle$ production events that miss the z cut would for $l_{\text{min}} \sim 10^{-19}$ metres be at most about 0.6, and for smaller values of \bar{L}_{loc} , this fraction would be smaller. Thus for comparison with the searches carried out so far at ATLAS and CMS, the order-of-magnitude estimate (84) does not require any correction for the z cut.

An early candidate for such modes was a 2.8 sigma bump at 1.8 TeV seen in the first 295 per nb of proton-proton collisions at 7 TeV centre-of-mass (c.o.m.) energy in ATLAS-CONF-2010-088 [127], which if real would have corresponded to a 27 pb cross-section for the modes to be produced in the s -channel and decay within the 15 + 15 centimetres along the beam line centred at the nominal interaction point (IP) allowed by the z cut on the primary vertex. However if the bump had been real there would have been a bump in the dijet final state with a similar total cross-section, and from Table II of [131], which used 1.0 per fb of proton-proton collisions at 7 TeV c.o.m. energy, the 95% CL upper limit on the total cross-section of such a bump at 1.8 TeV in the dijet final state is now about 0.1 pb.

In a recent search for narrow high-mass resonances decaying into e^+e^- or $\mu^+\mu^-$ final states in about 1.1 per fb of proton-proton collisions at 7 TeV c.o.m. energy, with each lepton having transverse momentum $p_T > 25$ GeV, ATLAS found no significant excess above the SM background in the search region from about 110 GeV to 2 TeV [129]. The signal acceptances were around 65% for electrons and 40% for muons, and from Figure 1 of this article, the SM background in the e^+e^- final state from about 120 GeV to 2 TeV is roughly:

$$\left. \frac{d\sigma}{d\left(\frac{m_{e^+e^-}}{\text{TeV}}\right)} \right|_{\text{ATLAS}} \simeq 6.0 \left(\frac{m_{e^+e^-}}{\text{TeV}} \right)^{-5.14} \text{fb}, \quad (88)$$

and the SM background in the $\mu^+\mu^-$ final state is roughly the same.

In a recent search for evidence of ADD large flat extra dimensions [1, 2, 3] in the $\mu^+\mu^-$ final state in about 1.2 per fb of proton-proton collisions at 7 TeV c.o.m. energy, with each muon having transverse momentum $p_T > 35$ GeV, CMS found no significant excess above the SM background in the search region from about 120 GeV to 3 TeV [69]. The simulated reconstruction efficiency for high mass Drell-Yan dimuon events in the selected acceptance range was above 90%, and from Figure 1 of this article, the SM background from about 120 GeV to 2 TeV is roughly:

$$\left. \frac{d\sigma}{d\left(\frac{m_{\mu^+\mu^-}}{\text{TeV}}\right)} \right|_{\text{CMS}} \simeq 4.4 \left(\frac{m_{\mu^+\mu^-}}{\text{TeV}} \right)^{-5.60} \text{fb}. \quad (89)$$

In a recent search for evidence of ADD or Randall-Sundrum extra dimensions [64] in the diphoton final state in 2.2 per fb of proton-proton collisions at 7 TeV c.o.m. energy, with each photon having transverse energy $E_T > 70$ GeV, CMS found no significant excess above the SM background in the search region from about 150 GeV to 2 TeV [132]. The corresponding diphoton reconstruction and identification efficiency was about 76%, and from Figure 1 of this article, the SM background from about 150 GeV to 2 TeV is roughly:

$$\left. \frac{d\sigma}{d\left(\frac{m_{\gamma\gamma}}{\text{TeV}}\right)} \right|_{\text{CMS}} \simeq 2.1 \left(\frac{m_{\gamma\gamma}}{\text{TeV}} \right)^{-5.31} \text{fb}. \quad (90)$$

In a recent search for evidence of ADD or Randall-Sundrum extra dimensions in the diphoton final state in 2.12 per fb of proton-proton collisions at 7 TeV c.o.m. energy, with each photon having transverse energy $E_T > 25$ GeV, ATLAS found no significant excess above the SM background in the search region from about 150 GeV to 2 TeV [72]. The selection efficiency for events within the detector acceptance was about 70%, and from Figure 1 of this article, the SM background from about 150 GeV to 2 TeV is roughly:

$$\left. \frac{d\sigma}{d\left(\frac{m_{\gamma\gamma}}{\text{TeV}}\right)} \right|_{\text{ATLAS}} = 4.3 \left(\frac{m_{\gamma\gamma}}{\text{TeV}} \right)^{-5.25} \text{fb}. \quad (91)$$

Let t denote one of the types of mode for which there might be a sufficiently large number of approximately degenerate modes of intrinsic mass $\simeq \bar{m}_t$ for them to produce a bump in the above cross-sections if they were produced in the s -channel at the LHC. Thus t denotes either the harmonic 3-forms C_{ABC} of intrinsic mass $\bar{m}_{3f} \simeq 0.2$, or the lightest generic classically massive modes of one of the types $C_{\mu\nu\sigma}$, $C_{\mu\nu A}$, $C_{\mu AB}$, $s_{\mu\nu}$,

h_{AA} , and $h_{\mu A}$, with classical intrinsic masses 3, 2, 1, 3, 3, and 2 respectively, or the lightest generic modes of t_{AB} , which from the discussion after (65) would be tachyonic unless the last two terms in (64) lift their squared masses sufficiently. The much larger intrinsic mass $\simeq 9$ calculated after (67) for the dilaton/radion mode of h_{AA} , which is classically massless, suggests that when the contributions of the $\Gamma_{\text{SG}}^{(8,\text{bos})}$ term (4) in (2) are included, the intrinsic masses of all modes other than the harmonic 3-forms, harmonic 2-forms, and harmonic 1-forms might be ~ 8 or more.

The harmonic 2-forms and harmonic 1-forms, and also any other non-generic modes, sometimes called supercurvature modes [120], such as the light modes in the far-from-isotropic closed hyperbolic 7-manifolds considered in the paragraph after (42), whose classical squared intrinsic masses are less than the minimum value of the classical squared intrinsic mass of the corresponding type of mode on uncompactified H^7 , are expected to be too few in number to be seen at the LHC.

If the actual number of approximately degenerate modes of type t and intrinsic mass $\simeq \bar{m}_t$ is $N_t = x_t N_{\text{Weyl}, \bar{m}_t}$, where x_t is expected from the discussion before (87) to be < 1 except for the harmonic 3-form modes, the estimated total cross-section for $\text{proton} + \text{proton} \rightarrow \text{any mode of type } t + X \rightarrow f + X$ is by (84) obtained from the reference estimate (87) by multiplying by x_t^2 . Thus for $f = e^+ + e^-$, the requirement that the total cross-section for this process, spread over a peak of width μ_t centred at $m_{e^+e^-} = m_t$, should be less than (88), gives on using $P = 3.5 \text{ TeV}$ and $1\text{fb} = 2.569 \times 10^{-6} \text{ TeV}^{-2}$:

$$x_t^2 \bar{m}_t^{18} \ln \frac{2.1 \text{ TeV}}{m_t} < 0.01 \frac{\mu_t^2}{m_t^2} \left(\frac{m_t}{\text{TeV}} \right)^{0.20} \quad (92)$$

in order of magnitude, which would have applied from about 120 GeV to 2 TeV if the statistics had been sufficient. However the total number of background events expected for $m_{e^+e^-} > 1 \text{ TeV}$ is only about 1, so the limit from [129] is weaker than (92) for $m_t > 1 \text{ TeV}$.

For the harmonic 3-forms, the reference estimate of the number N of modes is $N_{3\text{-form,ref}} = 0.06 \frac{\bar{V}_7}{\ln \bar{V}_7}$, which for $\bar{V}_7 \sim 10^{34}$ is $\simeq 4 \times 10^6 N_{\text{Weyl}, 0.2}$, so if the actual number of approximately degenerate harmonic 3-form modes of intrinsic mass $\bar{m}_{3f} \simeq 0.2$ is $N_{3f} = \tilde{x}_{3f} N_{3\text{-form,ref}}$, the limit (92) becomes:

$$\tilde{x}_{3f}^2 \ln \frac{2.1 \text{ TeV}}{m_{3f}} < 0.01 \frac{\mu_{3f}^2}{m_{3f}^2} \left(\frac{m_{3f}}{\text{TeV}} \right)^{0.20} \quad (93)$$

in order of magnitude.

The logarithmic factor in the left-hand sides of (92) and (93) decreases from about 3 at $m_t \simeq 120$ GeV to 0 at $m_t = 2.1$ TeV, so allowing for the low statistics for $m_t > 1$ TeV, the limits from the search [129] are that if m_{3f} or an m_t lies in the range from about 120 GeV to about 1.5 TeV, then the corresponding adjustment factor $\tilde{x}_{3f} = N_{3f}/N_{3\text{-form,ref}}$ or $x_t = N_t/N_{\text{Weyl},\bar{m}_t}$ is bounded in order of magnitude by:

$$\tilde{x}_{3f} < 0.1 \frac{\mu_{3f}}{m_{3f}}, \quad x_t \bar{m}_t^9 < 0.1 \frac{\mu_t}{m_t}. \quad (94)$$

The limits (94) give absolute bounds in order of magnitude on the adjustment factors \tilde{x}_{3f} and x_t if the corresponding mass m_{3f} or m_t lies in the range from about 120 GeV to about 1.5 TeV, since $\frac{\mu_t}{m_t} \leq 1$ in order of magnitude, and for t other than the harmonic 3-forms, the intrinsic mass \bar{m}_t seems likely to be larger than 1, and possibly as large as ~ 8 or more.

The backgrounds (89), (90), and (91) are equal in order of magnitude to the background (88) at corresponding final state masses m_f , and cover roughly the same range of m_f from about 120 GeV to about 2 TeV, and the corresponding searches have the same lack of statistics for m_f above 1 TeV as the search [129]. Thus the limits from the searches [69], [132], and [72] are also that if m_{3f} or an m_t lies in the range from about 120 GeV to about 1.5 TeV, then the corresponding adjustment factor \tilde{x}_{3f} or x_t is bounded in order of magnitude by (94).

If the order of magnitude bound $\tilde{x}_{3f} < 0.1$ was not satisfied, and \tilde{x}_{3f} was also sufficiently large for $m_{3f} < 120$ GeV to be excluded by earlier searches, for example at the Tevatron and LEP, then since $m_{3f} \simeq 0.2 \frac{A}{B}$ from subsection 2.1.1, where the constant A in the metric ansatz (14) lies between about 0.7 and 0.9, from the discussion following (38), and the best value of the constant B in the metric ansatz (14) is $B \simeq 0.28 \kappa_{11}^{2/9} \simeq 1.2 M_{11}^{-1}$, from subsection 1.1, $m_{3f} > 1.5$ TeV would imply $M_{11} > 10$ TeV, corresponding to $\kappa_{11}^{-2/9} > 2.3$ TeV, which is a stronger limit than the current experimental lower bound on M_{11} for 7 flat extra dimensions, which is roughly $M_{11} \geq 2.3 \pm 0.7$ TeV, corresponding to $\kappa_{11}^{-2/9} \geq 0.55 \pm 0.2$ TeV [65, 66, 67, 68, 69, 70, 71, 73, 74, 75].

If m_{3f} is smaller than about 300 GeV, and \tilde{x}_{3f} satisfies (94) if $m_{3f} > 120$ GeV, and is sufficiently small to have allowed the harmonic 3-form modes to have escaped discovery at the Tevatron, and at LEP if $m_{3f} < 209$ GeV, then m_t could be under 1.5 TeV for some of the other types of mode, if the second bound in (94) is satisfied for that t . From the discussion before (87), it seems likely that for t other than the harmonic 3-forms, a better estimate of the number N_t of modes than $N_{\text{Weyl},\bar{m}_t}$ might

be $N_{\text{Weyl}, \sqrt{\bar{m}_t^2 - \bar{m}_0^2}}$ for some $\bar{m}_0 < \bar{m}_t$, where $N_{\text{Weyl}, \bar{m}}$ was defined in (85). Then the second bound in (94) becomes:

$$\left(\bar{m}_t^2 - \bar{m}_0^2\right)^{\frac{7}{2}} \bar{m}_t^2 < 0.1 \frac{\mu_t}{m_t}. \quad (95)$$

This form of the bound cannot be used if t denotes the lightest generic modes of t_{AB} , which would be tachyonic unless the last two terms in (64) lift their squared masses sufficiently, but if t denotes the lightest generic classically massive modes of one of the remaining types $C_{\mu\nu\sigma}$, $C_{\mu\nu A}$, $C_{\mu AB}$, $s_{\mu\nu}$, h_{AA} , and $h_{\mu A}$, whose classical intrinsic masses $\sqrt{\bar{m}_t^2 - \bar{m}_0^2}$ lie in the range 1 to 3, then it implies that $m_t > 1.5$ TeV. If we then assume that for at least one of these types of mode, m_t is not much larger than the mass m_{dil} of the dilaton/radion mode of h_{AA} , which from the paragraph after (67) is $\simeq 9\frac{A}{B}$, we find $M_{11} > 0.2$ TeV, which corresponds to $\kappa_{11}^{-2/9} > 50$ GeV. This then implies $m_{3f} \simeq 0.2\frac{A}{B} > 30$ GeV. These limits do not depend on the value of \tilde{x}_{3f} .

The tachyonic \bar{m}^2 at the bottom of the \bar{m}^2 spectrum of the generic modes of t_{AB} , when the last two terms in (64) are neglected, is $\simeq -4$ from the discussion after (65), so if the last two terms in (64) contribute a term $\sim \bar{m}_{\text{dil}}^2 \simeq 80$ to \bar{m}^2 for t_{AB} , the lightest generic modes of t_{AB} will not be much lighter than the lightest generic classically massive modes of the other types other than C_{ABC} , so will also be heavier than around 1.5 TeV.

Thus it seems likely that if modes decaying along the beam line outside the interaction region are to be observable at the LHC with 7 TeV or 8 TeV c.o.m. energy, these modes must be the C_{ABC} harmonic 3-form modes whose mass was approximately calculated in subsection 2.1.1 as $m_{3f} \simeq 0.2\frac{A}{B}$, and whose number is $N_{3f} = \tilde{x}_{3f} N_{3\text{-form,ref}}$, where $N_{3\text{-form,ref}} = 0.06 \frac{\bar{V}_7}{\ln \bar{V}_7}$ from the discussion before (87), and \tilde{x}_{3f} satisfies the first bound in (94) in order of magnitude. These modes are pseudo-scalars along the extended dimensions, and were shown in subsection 2.1.2 to have axion-like couplings to the SM gauge bosons.

For these modes, the estimates in subsection 3.1 can be put on a slightly firmer foundation. Let \bar{H}^n be a closed hyperbolic n -manifold, $n \geq 2$, and \bar{H}^p be a closed p -manifold that is embedded as a minimal-area p -cycle in \bar{H}^n , where $1 \leq p \leq n-1$, and \bar{H}^p is closed hyperbolic for $p \geq 2$. Near \bar{H}^p we can choose the coordinates on \bar{H}^n to be $\bar{x}^A = (\hat{x}^a, \theta^i, \bar{y})$, where \hat{x}^a are coordinates on \bar{H}^p , θ^i are coordinates on the unit $(n-p-1)$ -sphere, and \bar{y} is the intrinsic geodesic distance from \bar{H}^p . The metric is:

$$ds_n^2 = B^2 \bar{g}_{AB} d\bar{x}^A d\bar{x}^B = B^2 \left(\cosh^2 \bar{y} \hat{g}_{ab} d\hat{x}^a d\hat{x}^b + \sinh^2 \bar{y} \check{g}_{ij} d\theta^i d\theta^j + d\bar{y}^2 \right), \quad (96)$$

where \hat{g}_{ab} is a metric on \bar{H}^p of sectional curvature -1 , and \check{g}_{ij} is a metric on the unit $(n - p - 1)$ -sphere.

Let $\omega_{A_1 \dots A_p}$ be a harmonic p -form on \bar{H}^n that coincides with the p -volume form on \bar{H}^p at $\bar{y} = 0$, and does not closely coincide with a nonzero multiple of the p -volume form on any other minimal-area p -cycle in \bar{H}^n . The integral $\int d\hat{x}^{a_1} \dots d\hat{x}^{a_p} \omega_{a_1 \dots a_p}(\hat{x}, \theta, \bar{y})$ at fixed θ^i and \bar{y} is independent of the θ^i and \bar{y} by the generalized Stokes's theorem [133], so for \bar{y} up to the smallest value at which a point of \bar{H}^n has two different representations in these coordinates, $\omega_{a_1 \dots a_p}(\hat{x}, \theta, \bar{y})$ will be approximately independent of the θ^i and \bar{y} . If \bar{H}^n has intrinsic diameter substantially larger than 1 and is reasonably isotropic, in the sense that it has an approximately spherical Dirichlet domain in n -dimensional hyperbolic space H^n , then $\omega_{a_1 \dots a_p}(\hat{x}, \theta, \bar{y})$ could be approximately independent of the θ^i and \bar{y} up to values of \bar{y} that are substantially larger than 1. In that case the integral $\int_{\bar{H}^n} d^n \bar{x} \sqrt{\bar{g}} \bar{g}^{A_1 B_1} \dots \bar{g}^{A_p B_p} \omega_{A_1 \dots A_p} \omega_{B_1 \dots B_p}$ will be approximately equal to the contribution to it from the region with \bar{y} less than about 2 or 3 if $2p > n - 1$, since the factor $e^{-2p\bar{y}}$ from the inverse metrics then outweighs the factor $e^{(n-1)\bar{y}}$ in $\sqrt{\bar{g}}$ for \bar{y} larger than about 1, so for $2p > n - 1$ the harmonic p -form $\omega_{A_1 \dots A_p}$ is effectively localized in a region of intrinsic half-thickness $\bar{y} \sim 1$ centred at \bar{H}^p .

The case $p = 3$, $n = 7$ is on the borderline where this form of geometric localization just fails to occur. If we convert the coordinate indices of $\omega_{A_1 A_2 A_3}$ to local Lorentz indices by contraction with a vielbein $\bar{e}^A_{\hat{B}}$, where hatted indices are local Lorentz indices and $\delta^{\hat{C}\hat{D}} \bar{e}^A_{\hat{C}} \bar{e}^B_{\hat{D}} = \bar{g}^{AB}$, then the coordinate scalar $\omega_{\hat{A}_1 \hat{A}_2 \hat{A}_3}$ has the same \bar{y} -dependence $e^{-3\bar{y}}$ for moderate $\bar{y} > 1$ as the amplitude of the spherically symmetric eigenmodes of the Laplace-Beltrami operator on uncompactified H^7 [123, 124]. However the effective rate of decrease of $\omega_{\hat{A}_1 \hat{A}_2 \hat{A}_3}$ with increasing \bar{y} is expected to be more rapid than $e^{-3\bar{y}}$ due to Anderson localization, which is an interference effect in which waves fail to propagate in a disordered medium, due to interference between multiple scattering paths [122, 134, 135, 136].

The Ioffe-Regel criterion for Anderson localization of single-particle wavefunctions in a disordered potential is that wavefunctions are localized when the mean free path between scatterings is smaller than the wavelength [122, 137]. The harmonic 3-forms are classically massless, so if \bar{H}^7 is reasonably isotropic, their intrinsic wavelength on \bar{H}^7 is roughly the intrinsic diameter \bar{L}_7 of \bar{H}^7 . The classical dynamics of a free particle in a compact hyperbolic space is strongly chaotic, and the Gutzwiller trace formula, which gives the semiclassical correspondence for classically chaotic systems and relates

a set of periodic orbits along closed geodesics to a set of energy eigenstates, becomes for compact hyperbolic spaces an exact relation known as the Selberg trace formula [120, 138, 139]. Thus it seems likely that both classically and quantum mechanically, the effective mean free path between scatterings on \bar{H}^7 will be at most \bar{L}_7 , so that harmonic 3-forms on \bar{H}^7 will behave roughly as $e^{-\left(3+\frac{1}{L_{\text{loc}}}\right)\bar{y}}$ for $\bar{y} > 1$, where $\bar{L}_{\text{loc}} > 0$ is the intrinsic localization length.

The intrinsic diameter \bar{L}_3 of a minimal-area 3-cycle \bar{H}^3 in \bar{H}^7 cannot be more than the intrinsic diameter \bar{L}_7 of \bar{H}^7 , so if \bar{H}^3 is reasonably isotropic, it cannot have intrinsic 3-volume \bar{V}_3 greater than $\sim e^{(3-1)\bar{L}_7/2}$, from page 9 of [20]. For $\bar{L}_7 \simeq 28$, from the discussion following (41), on page 14, this gives \bar{V}_3 not above $\sim 10^{12}$, so that the intrinsic 7-volume of the region of \bar{H}^7 within intrinsic distance $\bar{y} < 1$ from \bar{H}^3 is not above $\sim 10^{12}$, which is very small in comparison to the intrinsic 7-volume $\bar{V}_7 \sim 10^{35}$ of \bar{H}^7 , if $\kappa_{11}^{-2/9}$ is comparable to its current experimental lower limit. Thus for a rough first approximation we can treat \bar{H}^3 as a point, and \bar{y} as the intrinsic geodesic distance from that point, and to this approximation the behaviour $e^{-\left(3+\frac{1}{L_{\text{loc}}}\right)\bar{y}}$ of the harmonic 3-forms is the behaviour assumed in subsection 3.1.

The radio-frequency (RF) cavities that accelerate the protons in the LHC beams operate at 400 MHz, so the separation between adjacent RF “buckets” is 2.5 ns, which corresponds to a separation of 75 cm in the laboratory frame [140]. The r.m.s. length of the bunch of protons in a single RF bucket is 7.5 cm in the laboratory frame [140, 141], and during the 2011 proton-proton runs, one RF bucket in 20 was actually filled with a bunch, so the actual separation between adjacent bunches was 15 metres in the laboratory frame. This is also the planned separation between adjacent bunches for the 2012 proton-proton runs, for which the energy of a proton in one of the beams is to be 4 TeV [142].

From page 44 of [140], the r.m.s. beam radius at the interaction point (IP) of one of the two principal experiments was initially planned to be $16\mu\text{m}$, with the r.m.s. divergence of a beam at the IP set at $32\mu\text{rad}$, and the crossing angle set at $200\mu\text{rad}$. Thus the collisions would take place in the middle 7.5 cm of a beam crossing region of length about 32 cm, that is about $32\mu\text{m}$ in diameter at its centre, and tapers to a point at each end. ATLAS and CMS appear to use approximately these parameters [143, 144, 145, 141], except that from page 273 of [141], the crossing angle in CMS is $285\mu\text{rad}$, and from pages 2 to 3 of [146], the crossing angle in ATLAS might also be

285 μrad . Thus in both ATLAS and CMS, the collisions take place in an approximately cylindrical region of diameter $\simeq 32\mu\text{m}$ and length $\simeq 7.5$ cm centred at the IP, and the experiments must detect jets and charged leptons emitted from any point in this region, so as not to waste part of the available luminosity.

In practice during 2011 ATLAS appears to have imposed a cut requiring the distance $|z|$ along the beam line from the primary interaction vertex to the IP to be less than 20 cm for inclusive final states or final states containing muons, in order to reduce the background from cosmic ray muons [147, 148, 149, 128], and CMS has sometimes imposed a cut requiring $|z| < 12$ cm to reduce the background from cosmic ray muons [130], while for dijet final states, ATLAS does not appear to impose any cut on $|z|$ [150, 151, 152, 131], although in practice a limit of roughly $|z| < 6$ cm might arise from finding the event vertex or vertices using tracks that originate in the beam collision spot [153], since for 7 TeV c.o.m. energy, the z -distribution of primary interaction vertices is a Gaussian with $\sigma \simeq 2.2$ cm [154]. For a rough estimate at 7 TeV or 8 TeV c.o.m. energy, I shall treat the interaction region as extending for 6 cm in each direction along the beam line from the IP.

The ATLAS Inner Detector is 7 metres in length along the beam line [143], and the CMS Inner Tracking System is 5.4 metres in length along the beam line [141]. The central barrel part of the ATLAS Inner Detector is 1.6 metres in length, with the remainder of the length of the Inner Detector consisting of two identical end caps, and the CMS Tracker Inner Barrel is 1.3 metres in length, surrounded by the Tracker Outer Barrel which is 2.2 metres in length. From page 24 of [146], the ATLAS detector is capable of measuring the z values of tracks roughly perpendicular to the beam line up to at least $|z| = 1$ metre, and thus beyond the central barrel part of the ATLAS Inner Detector, and from page 3 of [130], CMS is capable of reconstructing tracks from decays that occur up to 50 cm from the beam line, although with significantly less than 100% efficiency. I shall assume that both ATLAS and CMS can approximately measure the z values of tracks roughly perpendicular to the beam line, over the whole length of their Inner Detector or Inner Tracking System, although with substantially less than 100% efficiency for finding tracks at the larger $|z|$ values.

For a reference estimate I shall consider the ATLAS Inner Detector, and thus consider modes that decay along the beam line at a distance between 6 cm and 3.5 metres from the IP. From (69) and (70), on page 32, the fraction of the harmonic 3-form modes, of intrinsic mass $\bar{m}_{3f} \simeq 0.2$, that decay further than a distance $z \gg l_{\min} \sim 10^{-19}$ metres

along the beam line from the IP, is approximately $\left(\frac{l_{\min}}{z}\right)^{\frac{1}{3\bar{L}_{\text{loc}}+1}} \sim \left(\frac{10^{-19} \text{ metres}}{z}\right)^{\frac{1}{3\bar{L}_{\text{loc}}+1}}$, for $\bar{L}_{\text{loc}} > 1$, where \bar{L}_{loc} is the intrinsic localization length of the harmonic 3-forms on \bar{H}^7 . For reasonably isotropic \bar{H}^7 , whose intrinsic diameter \bar{L}_7 would from the discussion following (41), on page 14, be about 28, if $\kappa_{11}^{-2/9}$ and M_{11} are close to their current experimental lower limits, for 7 flat extra dimensions, of about 0.55 TeV and 2.3 TeV respectively, \bar{L}_{loc} would be expected, from the above discussion of Anderson localization, to be somewhere in the range from about 4 to about 28. Let

$$f_{0.06,3.5}(\bar{L}_{\text{loc}}) \equiv \left(\frac{10^{-19} \text{ metres}}{0.06 \text{ metres}}\right)^{\frac{1}{3\bar{L}_{\text{loc}}+1}} - \left(\frac{10^{-19} \text{ metres}}{3.5 \text{ metres}}\right)^{\frac{1}{3\bar{L}_{\text{loc}}+1}} \quad (97)$$

be the fraction of the harmonic 3-form modes that decay between 6 cm and 3.5 metres along the beam line from the IP. We then find the values:

\bar{L}_{loc}	1	2	4	8	12	16	20	24	28
$f_{0.06,3.5}$	0.000023	0.0013	0.012	0.029	0.034	0.035	0.033	0.031	0.029

Thus for \bar{L}_{loc} throughout most of the expected range, $f_{0.06,3.5}(\bar{L}_{\text{loc}}) \simeq 0.03$, and this is valid within a factor of 3 throughout the whole expected range. Thus from (87) and the following discussion, on page 38, with $\bar{m} = 0.2$, and the discussion around (92) and (93), on page 41, we obtain:

$$\sigma(\text{prot} + \text{prot} \rightarrow \text{any } n + X \rightarrow f + X)_{0.06,3.5} \sim 10^{-5} \frac{\tilde{x}_{3f}^2}{\mu_{3f} m_{3f}} \left(\frac{P}{m_{3f}}\right)^{2.34} \ln \frac{0.6P}{m_{3f}}, \quad (98)$$

as an order of magnitude estimate of the cross section for a harmonic 3-form mode of intrinsic mass $\bar{m}_{3f} \simeq 0.2$ and mass $m_{3f} = \bar{m}_{3f} \frac{A}{B} < 0.6P$ to be produced in the s -channel and decay between 0.6 cm and 3.5 metres along the beamline from the IP, where the number of approximately degenerate harmonic 3-form modes of intrinsic mass $\bar{m}_{3f} \simeq 0.2$ is $N_{3f} = \tilde{x}_{3f} N_{3\text{-form,ref}} = \tilde{x}_{3f} 0.06 \frac{\bar{V}_7}{\ln \bar{V}_7}$, P is the energy per proton, which was 3.5 TeV at the LHC in 2011, and is to be 4.0 TeV at the LHC in 2012 [142], the warp factor A lies between about 0.7 and 0.9, from the discussion between (38), on page 13, and (39), on page 14, the curvature radius B of \bar{H}^7 is $B \simeq 0.28 \kappa_{11}^{2/9}$, from subsection 1.1, starting on page 9, and if m_{3f} lies in the range from about 120 GeV to about 1.5 TeV, then \tilde{x}_{3f} is bounded in order of magnitude by $\tilde{x}_{3f} < 0.1 \frac{\mu_{3f}}{m_{3f}}$, from (94), on page 42. μ_{3f} is the width of the distribution of the masses of the harmonic 3-form modes, which was assumed in subsection 3.2, starting on page 33, to be $\geq \Gamma_n$ for all the harmonic 3-form modes $|n\rangle$, where Γ_n is the total width of the mode $|n\rangle$, in order to derive the total cross-section estimate (87), on page 38.

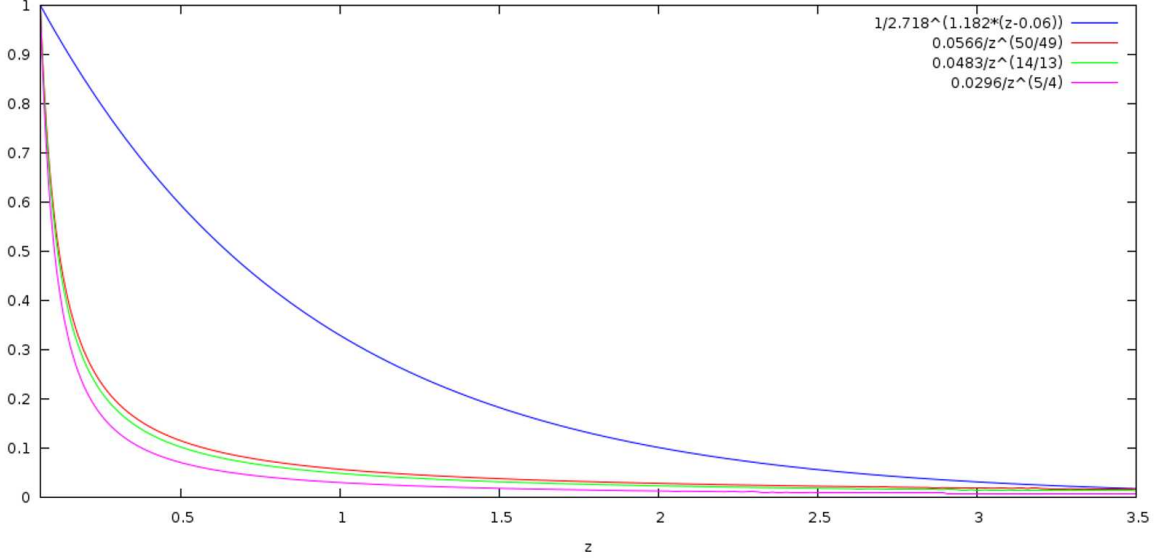


Figure 1: Comparison of exponential and power dependence on z , for $z \geq 0.06$

Using $1 \text{ TeV}^{-2} = 0.3893 \text{ nb}$ and the limit (94), (98) becomes:

$$\sigma(\text{prot} + \text{prot} \rightarrow \text{any } n + X \rightarrow f + X)_{0.06, 3.5} < \frac{\mu_{3f}}{m_{3f}} \left(\frac{m_{3f}}{\text{TeV}} \right)^{-4.34} \text{ fb}, \quad (99)$$

as an order of magnitude upper limit on the cross-section for a harmonic 3-form mode of mass $\simeq m_{3f}$ to be produced in the s -channel and decay between 6 cm and 3.5 metres along the beam line from the IP, for $P = 3.5$ or 4 TeV per proton, and m_{3f} between about 120 GeV and 1.5 TeV. Thus if \tilde{x}_{3f} is at the upper limit allowed by (94), and $\mu_{3f} \sim m_{3f}$, then at the LHC design luminosity of 10 per nb per second [140], there would be about 0.1 such events per second if m_{3f} is 120 GeV, and about 10^{-6} such events per second if m_{3f} is 1.5 TeV, and if the LHC delivers the expected 15 to 19 per fb to ATLAS and CMS during 2012 [142], there would be about 10^5 such events in ATLAS and CMS during 2012 if m_{3f} is 120 GeV, and about 1 such event in ATLAS and CMS during 2012 if m_{3f} is 1.5 TeV.

Figure 1 shows the z -dependence $z^{-\frac{3\bar{L}_{\text{loc}}+2}{3\bar{L}_{\text{loc}}+1}}$ of the number of harmonic 3-form modes decaying between distances z and $z+dz$ along the beam line from the IP, for $6 \text{ cm} \leq z \leq 3.5 \text{ metres}$ and fixed dz , normalized to 1 at $z = 6 \text{ cm}$, for $\bar{L}_{\text{loc}} = 16, 4$, and 1, together with an exponential curve that matches the limiting case of large \bar{L}_{loc} at $z = 6 \text{ cm}$ and $z = 3.5 \text{ metres}$. This figure shows that the power-law z -dependence could be distinguished from a single exponential with a relatively small number of events, but it could be difficult to distinguish different values of \bar{L}_{loc} in the relevant range of about

$4 \leq \bar{L}_{\text{loc}} \leq 28$, even with the $\sim 10^5$ events expected during 2012 if m_{3f} is 120 GeV.

The principal backgrounds to this process are beam-induced backgrounds and cosmic-ray showers [146]. Beam-induced backgrounds are due to proton losses upstream of the IP. These result in cascades of secondary particles that fly through the detectors almost parallel to the beam line. The cosmic-ray showers are produced by cosmic rays, mostly protons and heavier nuclei, colliding with atoms in the Earth's atmosphere, and muons produced in these showers can penetrate down to the ATLAS and CMS detectors, which are situated in caverns about 100 metres underground [155]. The cosmic ray muons that reach ATLAS come mostly from above, and arrive mainly via two large access shafts that were used for the detector installation [156].

The harmonic 3-form modes are pseudo-scalars along the 3+1 extended dimensions, so their decay is isotropic in their rest frame. Their decay products will be boosted in the direction away from the IP in the laboratory frame, so the background from both beam-induced backgrounds and cosmic-ray muons could be reduced by selecting events where at least 2 charged leptons or 2 jets originate from a primary vertex that is at least 6 cm from the IP along the beam line, but within a few mm of the beam line in the transverse directions, with no missing transverse momentum, and a significant net longitudinal momentum in the direction away from the IP.

The initial, hardware-based stages of the ATLAS and CMS trigger systems use information only from the muon systems and calorimeters, so they accept events of this type. Approximate track reconstruction is not carried out until the later, software-based stages of the trigger systems, which can use the high-resolution position data from the inner detectors, in addition to the data from the muon systems and calorimeters [143, 141, 154, 157]. From the discussion before equation (97) above, ATLAS and CMS are able to reconstruct approximately the tracks from primary interaction vertices up to around 50 cm to 1 metre from the IP along the beam line, and their high-level triggers can accept and store these events for offline analysis. If sufficient rejection of the beam-induced backgrounds and the cosmic ray background could be achieved without reducing the signal too much, and \tilde{x}_{3f} is at the upper limit allowed by (94), and $\mu_{3f} \sim m_{3f}$, then the order of magnitude estimate (99) suggests that a 5-sigma discovery of the harmonic 3-form modes decaying more than 6 cm along the beam line from the IP could be achieved in 2012, if their central mass m_{3f} is not more than about 900 GeV, which corresponds roughly to $\kappa_{11}^{-2/9} < 1.6$ TeV and $M_{11} < 7$ TeV.

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References

- [1] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, “The Hierarchy Problem and New Dimensions at a Millimeter,” *Phys. Lett. B* **429** (1998) 263-272, arXiv:hep-ph/9803315.
- [2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, “New dimensions at a millimeter to a Fermi and superstrings at a TeV,” *Phys. Lett. B* **436** (1998) 257 - 263, arXiv:hep-ph/9804398.
- [3] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, “Phenomenology, Astrophysics and Cosmology of Theories with Sub-Millimeter Dimensions and TeV Scale Quantum Gravity,” *Phys. Rev. D* **59** (1999) 086004, arXiv:hep-ph/9807344.
- [4] E. Cremmer, B. Julia and J. Scherk, “Supergravity theory in 11 dimensions,” *Phys. Lett. B* **76** (1978) 409-412. Scanned version from KEK:
http://ccdb4fs.kek.jp/cgi-bin/img_index?7805106

- [5] P. Hořava and E. Witten, “Heterotic And Type I String Dynamics From Eleven Dimensions,” Nucl. Phys. **B460** (1996) 506-524, arXiv:hep-th/9510209.
- [6] P. Hořava and E. Witten, “Eleven-Dimensional Supergravity on a Manifold with Boundary,” Nucl. Phys. **B475** (1996) 94-114, arXiv:hep-th/9603142.
- [7] E. Witten, “Strong Coupling Expansion Of Calabi-Yau Compactification,” Nucl. Phys. **B471** (1996) 135-158, arXiv:hep-th/9602070.
- [8] I. G. Moss, “Boundary terms for eleven-dimensional supergravity and M-theory,” Phys. Lett. B **577** (2003) 71-75, arXiv:hep-th/0308159.
- [9] I. G. Moss, “Boundary terms for supergravity and heterotic M-theory,” Nucl. Phys. B **729** (2005) 179-202, arXiv:hep-th/0403106.
- [10] I. G. Moss, “A new look at anomaly cancellation in heterotic M-theory,” Phys. Lett. B **637** (2006) 93-96, arXiv:hep-th/0508227.
- [11] I. G. Moss, “Higher order terms in an improved heterotic M theory,” JHEP **0811** (2008) 067, arXiv:0810.1662 [hep-th].
- [12] N. Kaloper, J. March-Russell, G. D. Starkman, M. Trodden, “Compact hyperbolic extra dimensions: Branes, Kaluza-Klein modes and cosmology,” Phys. Rev. Lett. **85** (2000) 928-931, arXiv:hep-ph/0002001.
- [13] G. D. Mostow, “Quasi-conformal mappings in n -space and the rigidity of the hyperbolic space forms,” Publ. Math. IHES **34** (1968) 53-104.
- [14] G. D. Mostow, “Strong rigidity of locally symmetric spaces,” Ann. of Math. Studies, **78** (1973) 1-195.
- [15] G. Prasad, “Strong rigidity of rank 1 lattices,” Invent. Math. **21** (1973) 255 - 286.
- [16] William Thurston, The geometry and topology of 3-manifolds, Princeton University lecture notes (1978-1981).
<http://www.msri.org/publications/books/gt3m/>
- [17] M. Gromov, “Hyperbolic manifolds according to Thurston and Jørgensen,” Séminaire Bourbaki, **32e année**, **546** (1979/80) 40-53.
[http://www.ihes.fr/~gromov/PDF/1\[29\].pdf](http://www.ihes.fr/~gromov/PDF/1[29].pdf)

- [18] T. Gelander, “Homotopy type and volume of locally symmetric manifolds,” arXiv:math.GR/0111165.
- [19] M. Burger, T. Gelander, A. Lubotzky, and S. Mozes, “Counting hyperbolic manifolds,” Geometric and Functional Analysis **12** (2002) 1161-1173.
[http://www.ma.huji.ac.il/~alexclub/PAPERS/Counting hyperbolic manifolds/Counting hyperbolic manifolds.pdf](http://www.ma.huji.ac.il/~alexclub/PAPERS/Counting%20hyperbolic%20manifolds/Counting%20hyperbolic%20manifolds.pdf)
- [20] C. Austin, “ $d = 11$ supergravity on almost flat \mathbf{R}^4 times a compact hyperbolic 7-manifold, and the dip and bump seen in ATLAS-CONF-2010-088,” arXiv:1103.2732 [hep-th].
- [21] A. Lukas, B. A. Ovrut and D. Waldram, “On the four-dimensional effective action of strongly coupled heterotic string theory,” Nucl. Phys. B **532** (1998) 43-82, arXiv:hep-th/9710208.
- [22] R. Rohm, E. Witten, “The Antisymmetric Tensor Field in Superstring Theory,” Annals Phys. **170** (1986) 454-489.
- [23] E. Witten, “On flux quantization in M-theory and the effective action,” J. Geom. Phys. **22** (1997) 1-13, arXiv:hep-th/9609122.
- [24] A. A. Tseytlin, “ R^4 terms in 11 dimensions and conformal anomaly of $(2, 0)$ theory,” Nucl. Phys. **B584** (2000) 233-250, arXiv:hep-th/0005072.
- [25] M. B. Green and J. H. Schwarz, “Anomaly Cancellation In Supersymmetric $D = 10$ Gauge Theory And Superstring Theory,” Phys. Lett. B **149** (1984) 117-122. Scanned version from KEK:
http://ccdb4fs.kek.jp/cgi-bin/img_index?8412338
- [26] M. J. Duff, J. T. Liu and R. Minasian, “Eleven-dimensional origin of string / string duality: A one-loop test,” Nucl. Phys. B **452** (1995) 261-282, arXiv:hep-th/9506126.
- [27] E. Witten, “Five-brane effective action in M-theory,” J. Geom. Phys. **22** (1997) 103-133, arXiv:hep-th/9610234.
- [28] D. Freed, J. A. Harvey, R. Minasian and G. W. Moore, “Gravitational anomaly cancellation for M-theory fivebranes,” Adv. Theor. Math. Phys. **2** (1998) 601-618, arXiv:hep-th/9803205.

- [29] A. Bilal and S. Metzger, “Anomaly cancellation in M-theory: A critical review,” Nucl. Phys. **B675** (2003) 416-446, arXiv:hep-th/0307152.
- [30] J. A. Harvey, “TASI 2003 lectures on anomalies,” arXiv:hep-th/0509097.
- [31] S. P. de Alwis, “Anomaly cancellation in M-theory,” Phys. Lett. B **392** (1997) 332-334, arXiv:hep-th/9609211.
- [32] J. O. Conrad, “Brane tensions and coupling constants from within M-theory,” Phys. Lett. B **421** (1998) 119-124, arXiv:hep-th/9708031.
- [33] M. Faux, D. Lüst, B. A. Ovrut, “Intersecting orbifold planes and local anomaly cancellation in M theory,” Nucl. Phys. **B554** (1999) 437-483, arXiv:hep-th/9903028.
- [34] J. X. Lu, “Remarks on M theory coupling constants and M-brane tension quantizations,” arXiv:hep-th/9711014.
- [35] A. Bilal, J.-P. Derendinger, and R. Sauser, “M-Theory on S^1/\mathbf{Z}_2 : new facts from a careful analysis,” Nucl. Phys. B **576** (2000) 347-374, arXiv:hep-th/9912150.
- [36] T. Harmark, “Coupling constants and brane tensions from anomaly cancellation in M theory,” Phys. Lett. **B431** (1998) 295-302, arXiv:hep-th/9802190.
- [37] K. A. Meissner, M. Olechowski, “Anomaly cancellation in M theory on orbifolds,” Nucl. Phys. **B590** (2000) 161-172, arXiv:hep-th/0003233.
- [38] Y. Hyakutake and S. Ogushi, “Higher derivative corrections to eleven dimensional supergravity via local supersymmetry,” JHEP **0602** (2006) 068, arXiv:hep-th/0601092.
- [39] Y. Hyakutake, “Toward the determination of R^3F^2 terms in M -theory,” Prog. Theor. Phys. **118** (2007) 109, arXiv:hep-th/0703154.
- [40] R. R. Metsaev, “Eleven dimensional supergravity in light cone gauge,” Phys. Rev. **D71** (2005) 085017, arXiv:hep-th/0410239.
- [41] D. M. Richards, “The One-Loop Five-Graviton Amplitude and the Effective Action,” JHEP **0810** (2008) 042, arXiv:0807.2421 [hep-th].

- [42] D. M. Richards, “The One-Loop H^2R^3 and $H^2(DH)^2R$ Terms in the Effective Action,” JHEP **0810** (2008) 043, arXiv:0807.3453 [hep-th].
- [43] Y. Hyakutake and S. Ogushi, “ R^4 corrections to eleven dimensional supergravity via supersymmetry,” Phys. Rev. **D74** (2006) 025022, arXiv:hep-th/0508204.
- [44] Y. Hyakutake, “Higher derivative corrections in M-theory via local supersymmetry,” arXiv:0710.2673 [hep-th].
- [45] G. ’t Hooft and M. Veltman, “DIAGRAMMAR,” CERN report 73-9 (1973), reprinted in G. t Hooft, *Under the Spell of Gauge Principle*, World Scientific, Singapore (1994). <http://cdsweb.cern.ch/record/186259/files/p1.pdf>
- [46] B.W. Lee, “Gauge theories,” in *Les Houches 1975: Methods in Field Theory*, R. Balian and J. Zinn-Justin (Eds.), Elsevier, Amsterdam, 1976, p. 79.
- [47] A. A. Tseytlin, “Ambiguity in the Effective Action in String Theories,” Phys. Lett. B **176** (1986) 92-98. Scanned version from KEK:
http://ccdb4fs.kek.jp/cgi-bin/img_index?8702274
- [48] P. M. Stevenson, “Optimized Perturbation Theory,” Phys. Rev. **D23** (1981) 2916-2944.
- [49] S. Deser and D. Seminara, “Counterterms/M-theory corrections to D = 11 supergravity,” Phys. Rev. Lett. **82** (1999) 2435-2438, arXiv:hep-th/9812136.
- [50] S. Deser and D. Seminara, “Tree amplitudes and two-loop counterterms in D = 11 supergravity,” Phys. Rev. **D62** (2000) 084010, arXiv:hep-th/0002241.
- [51] K. Peeters, J. Plefka, and S. Stern, “Higher-derivative gauge field terms in the M-theory action,” JHEP **0508** (2005) 095, arXiv:hep-th/0507178.
- [52] A. Kehagias and H. Partouche, “On the exact quartic effective action for the type iib superstring,” Phys. Lett. B **422** (1998) 109, arXiv:hep-th/9710023.
- [53] D. J. Gross and J. H. Sloan, “The Quartic Effective Action for the Heterotic String,” Nucl. Phys. **B291** (1987) 41-89. Scanned version from KEK:
http://ccdb4fs.kek.jp/cgi-bin/img_index?200033932

- [54] R.S. Palais, “The principle of symmetric criticality,” *Comm. Math. Physics* **69** (1979) 19-30. <http://projecteuclid.org/euclid.cmp/1103905401>
- [55] S. Deser, J. Franklin, and B. Tekin, “Shortcuts to spherically symmetric solutions: A Cautionary note,” *Class. Quant. Grav.* **21** (2004) 5295-5296, [arXiv:gr-qc/0404120](https://arxiv.org/abs/gr-qc/0404120).
- [56] C. G. Torre, “Symmetric Criticality in Classical Field Theory,” [arXiv:1011.3429](https://arxiv.org/abs/1011.3429) [math-ph].
- [57] G. F. Giudice, R. Rattazzi and J. D. Wells, “Quantum gravity and extra dimensions at high-energy colliders,” *Nucl. Phys.* **B544** (1999) 3-38, [arXiv:hep-ph/9811291](https://arxiv.org/abs/hep-ph/9811291).
- [58] M. B. Green, J. Schwarz and E. Witten, *Superstring theory*, Vol. 1: *Introduction*, Vol. 2: *Loop amplitudes, anomalies and phenomenology*, Cambridge University Press, 1987.
- [59] C. Austin, “TeV-scale gravity in Hořava-Witten theory on a compact complex hyperbolic threefold,” [arXiv:0704.1476](https://arxiv.org/abs/0704.1476) [hep-th].
- [60] W. Israel, “Singular hypersurfaces and thin shells in general relativity,” *Nuovo Cim.* **B44** (1966) 1. Erratum: *Nuovo Cim.* **B48**, (1967) 463.
- [61] H. A. Chamblin and H. S. Reall, “Dynamic dilatonic domain walls,” *Nucl. Phys.* **B562** (1999) 133-157, [arXiv:hep-th/9903225](https://arxiv.org/abs/hep-th/9903225).
- [62] E. Dyer and K. Hinterbichler, “Boundary Terms, Variational Principles and Higher Derivative Modified Gravity,” *Phys. Rev.* **D79** (2009) 024028, [arXiv:0809.4033](https://arxiv.org/abs/0809.4033) [gr-qc].
- [63] K. Nakamura et al., (the Particle Data Group), “The Review of Particle Physics,” *J. Phys. G* **37** (2010) 075021. <http://pdg.lbl.gov/>
- [64] L. Randall and R. Sundrum, “A large mass hierarchy from a small extra dimension,” *Phys. Rev. Lett.* **83** (1999) 3370-3373, [arXiv:hep-ph/9905221](https://arxiv.org/abs/hep-ph/9905221).
- [65] R. Franceschini, G.F. Giudice, P.P. Giardino, P. Lodone, and A. Strumia, “LHC bounds on large extra dimensions,” *JHEP* **1105** (2011) 092, [arXiv:1101.4919v3](https://arxiv.org/abs/1101.4919v3) [hep-ph].

- [66] S. Chatrchyan *et al.* [CMS Collaboration], “Search for Large Extra Dimensions in the Diphoton Final State at the Large Hadron Collider,” JHEP **1105** (2011) 085, arXiv:1103.4279 [hep-ex].
- [67] The ATLAS Collaboration, “Search for New Phenomena in Monojet plus Missing Transverse Momentum Final States using 1 fb^{-1} of pp Collisions at $\sqrt{s} = 7 \text{ TeV}$ with the ATLAS Detector,” ATLAS-CONF-2011-096, 21 July 2011.
<http://cdsweb.cern.ch/record/1369187/files/ATLAS-CONF-2011-096.pdf>
- [68] The CMS Collaboration, “Search for Extra Dimensions in the Diphoton Final State with $0.9/\text{fb}$ of pp Collisions at the Large Hadron Collider,” CMS PAS EXO-11-038, 23 August 2011.
<http://cdsweb.cern.ch/record/1376706/files/EXO-11-038-pas.pdf>
- [69] The CMS Collaboration, “Search for Extra Dimensions in Dimuon Events in pp Collisions at $\sqrt{s} = 7 \text{ TeV}$,” CMS PAS EXO-11-039, 23 August 2011.
<http://cdsweb.cern.ch/record/1376670/files/EXO-11-039-pas.pdf>
- [70] The CMS Collaboration, “Search for ADD Extra-dimensions in Monophotons,” CMS PAS EXO-11-058, 25 August 2011.
<http://cdsweb.cern.ch/record/1377334/files/EXO-11-058-pas.pdf>
- [71] S. Chatrchyan *et al.* [CMS Collaboration], “Search for signatures of extra dimensions in the diphoton mass spectrum at the Large Hadron Collider,” arXiv:1112.0688 [hep-ex].
- [72] G. Aad *et al.* [ATLAS Collaboration], “Search for Extra Dimensions using diphoton events in 7 TeV proton-proton collisions with the ATLAS detector,” arXiv:1112.2194 [hep-ex].
- [73] S. Chatrchyan *et al.* [CMS Collaboration], “Search for microscopic black holes in pp collisions at $\sqrt{s} = 7 \text{ TeV}$,” arXiv:1202.6396 [hep-ex].
- [74] The ATLAS Collaboration, “Search for strong gravity effects in same-sign dimuon final states,” ATLAS-CONF-2011-065, 20 Apr 2011.
<http://cdsweb.cern.ch/record/1346080/files/ATLAS-CONF-2011-065.pdf>

- [75] The ATLAS Collaboration, “Search for Microscopic Black Holes in Multi-Jet Final States with the ATLAS Detector at $\sqrt{s} = 7$ TeV,” ATLAS-CONF-2011-068, 1 November 2011.
<http://cdsweb.cern.ch/record/1349309/files/ATLAS-CONF-2011-068.pdf>
- [76] P. Sarnak, “Selberg’s Eigenvalue Conjecture,” Notices of the AMS, **42** (1995) 1272-1277. <http://www.ams.org/notices/199511/sarnak.pdf>
- [77] S.-T. Yau, “Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold,” Ann. Scient. École Norm. Sup. **8** (1975) 487-507.
- [78] S. Agmon, “On the spectral theory of the Laplacian on noncompact hyperbolic manifolds,” Journées Équations aux dérivées partielles (1987) 1-16.
http://www.numdam.org/item?id=JEDP_1987____A17_0
- [79] R. Brooks and E. Makover, “The first eigenvalue of a Riemann surface,” Electronic Research Announcements of the American Mathematical Society **5** (1999) 76-81.
<http://www.ams.org/journals/era/1999-05-11/S1079-6762-99-00064-5/>
- [80] R. Brooks and E. Makover, “Riemann surfaces with large first eigenvalue,” Journal d’Analyse Mathématique **83** (2001) 243-258.
- [81] R. Brooks and E. Makover, “Belyi surfaces,” IMCP **15** (2001) 37-46.
- [82] R. Brooks and E. Makover, “Random Construction of Riemann Surfaces,” J. Differential Geom. **68** (2004) 121-157, arXiv:math/0106251.
- [83] Jeff Cheeger, “A lower bound for the smallest eigenvalue of the Laplacian,” in *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pp. 195-199, Princeton Univ. Press, Princeton, 1970.
- [84] http://en.wikipedia.org/wiki/Cheeger_constant
- [85] H. Donnelly, “The differential form spectrum of hyperbolic space,” Manuscripta Math. **33** (1981) 365-385.
- [86] P. Sarnak, “Arithmetic and geometry of some hyperbolic three manifolds,” Acta Mathematica **151** (1983) 253-295.

- [87] W. Luo, Z. Rudnick, and P. Sarnak, “On the generalized Ramanujan Conjectures for $GL(n)$,” *Proc. Symp. Pure Math.* **66-2** (1999) 301-311.
- [88] S.-y. Koyama, “The First Eigenvalue Problem and Tensor Products of Zeta Functions,” *Proceedings of the Japan Academy, Ser. A, Mathematical Sciences* **80** (2004-05) 35-39.
- [89] D. Orlando and S. C. Park, “Compact hyperbolic extra dimensions: a M-theory solution and its implications for the LHC,” *JHEP* **1008** (2010) 006, arXiv:1006.1901 [hep-th].
- [90] R. Mazzeo and R.S. Phillips, “Hodge theory on hyperbolic manifolds,” *Duke Math. J.* **60** (1990), 509-559.
- [91] H. Donnelly and F. Xavier, “On the Differential Form Spectrum of Negatively Curved Riemannian Manifolds,” *American Journal of Mathematics* **106** (1984) 169-185.
http://www.nd.edu/~fxavier/Publications/Donnelly_Xavier_84.pdf
- [92] B. Colbois and G. Courtois, “Les valeurs propres inférieures à $\frac{1}{4}$ des surfaces de Riemann de petit rayon d’injectivité,” *Comment. Math. Helv.* **64** (1989) 349-362.
- [93] B. Colbois and G. Courtois, “Convergence de variétés et convergence du spectre du Laplacien,” *Ann. Sci. École Norm. Sup.* **24** (1991) 507-518.
- [94] M. Gromov and I. Piatetski-Shapiro, “Non-arithmetic groups in Lobachevsky spaces,” *Inst. Hautes Études Sci. Publ.* **66** (1988) 93-103.
http://archive.numdam.org/article/PMIHES_1987__66__93_0.pdf
- [95] A. Proca, “Sur la théorie ondulatoire des électrons positifs et négatifs,” *J. Phys. Radium* **7** (1936) 347.
- [96] A. Proca, “Sur la théorie du positon,” *C. R. Acad. Sci. Paris* **202** (1936) 1366.
- [97] M. Gromov, “Volume and bounded cohomology,” *Publ. Math. IHÉS* **56** (1982), 5-100. http://archive.numdam.org/article/PMIHES_1982__56__5_0.pdf
- [98] M. Gromov, theorem 2 in W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of Nonpositive Curvature*, Birkhauser, 1985.

- [99] W. Lück, “Approximating L^2 -invariants by their finite dimensional analogues,” *Geom. and Func. Anal.*, **4** (1994) 455-481.
<http://wwwmath.uni-muenster.de/users/lueck/publ/lueck/r.pdf>
- [100] B. Clair and K. Whyte, “Growth of Betti Numbers,” *Topology* **42** (2003) 1125-1142, [arXiv:math/0111120](https://arxiv.org/abs/math/0111120) [math.GT].
- [101] X. Xue, “On the Betti numbers of a hyperbolic manifold,” *Geometric And Functional Analysis* **2** (1992) 126-136.
- [102] C. Beasley and E. Witten, “A Note on fluxes and superpotentials in M theory compactifications on manifolds of G(2) holonomy,” *JHEP* **0207** (2002) 046, [arXiv:hep-th/0203061](https://arxiv.org/abs/hep-th/0203061).
- [103] S. Weinberg, “A New Light Boson?,” *Phys. Rev. Lett.* **40** (1978) 223-226.
- [104] F. Wilczek, “Problem of Strong p and t Invariance in the Presence of Instantons,” *Phys. Rev. Lett.* **40** (1978) 279-282. Scanned version from KEK:
http://ccdb5fs.kek.jp/cgi-bin/img_index?197801210
- [105] R.H. Dalitz, “CXII. On the analysis of τ -meson data and the nature of the τ -meson,” *Phil. Mag.* **44** (1953) 1068-1080.
- [106] E. Fabri, “A study of τ -meson decay,” *Nuovo Cim.* **11** (1954) 479-491.
- [107] D.H. Perkins, *Introduction to High Energy Physics*, third edition, Addison-Wesley Publishing Company, Inc., 1987.
- [108] S. M. Christensen and M. J. Duff, “Quantizing Gravity with a Cosmological Constant,” *Nucl. Phys. B* **170** (1980) 480-506.
- [109] S. Randjbar-Daemi, A. Salam and J. A. Strathdee, “Towards A Selfconsistent Computation Of Vacuum Energy In Eleven-dimensional Supergravity,” *Nuovo Cim. B* **84** (1984) 167. Scanned version from KEK:
http://ccdb5fs.kek.jp/cgi-bin/img_index?8407366
Scanned version from ICTP:
<http://library.ictp.trieste.it/DOCS/P/84/021.pdf>
- [110] D. Hoover and C. P. Burgess, “Ultraviolet sensitivity in higher dimensions,” *JHEP* **0601** (2006) 058, [arXiv:hep-th/0507293](https://arxiv.org/abs/hep-th/0507293).

- [111] R. Bousso, J. Polchinski, “Quantization of four form fluxes and dynamical neutralization of the cosmological constant,” JHEP **0006** (2000) 006, arXiv:hep-th/0004134.
- [112] A. G. Riess *et al.* [Supernova Search Team], “Observational evidence from supernovae for an accelerating universe and a cosmological constant,” Astron. J. **116** (1998) 1009-1038, arXiv:astro-ph/9805201.
- [113] P. M. Garnavich *et al.* [Supernova Search Team Collaboration], “Supernova Limits on the Cosmic Equation of State,” Astrophys. J. **509** (1998) 74-79, arXiv:astro-ph/9806396.
- [114] S. Perlmutter *et al.* [Supernova Cosmology Project], “Measurements of omega and lambda from 42 high redshift supernovae,” Astrophys. J. **517** (1999) 565-586, arXiv:astro-ph/9812133.
- [115] http://en.wikipedia.org/wiki/Jordan_and_Einstein_frames
- [116] M. J. Duff, B. E. W. Nilsson and C. N. Pope, “The Criterion For Vacuum Stability In Kaluza-Klein Supergravity,” Phys. Lett. B **139** (1984) 154-158.
- [117] E. Delay, “Essential spectrum of the Lichnerowicz Laplacian on two-tensors on asymptotically hyperbolic manifolds,” Journal of Geometry and Physics **43** (2002) 33-44.
http://www.univ-avignon.fr/fileadmin/documents/Users/Fiches_X_P/Delay/spectrum3.pdf
- [118] J. M. Lee, “Fredholm operators and Einstein metrics on conformally compact manifolds,” Mem. Amer. Math. Soc. **183** (2006), 83 + vi pages, arXiv:math.DG/0105046.
- [119] M. Fierz and W. Pauli, “On relativistic wave equations for particles of arbitrary spin in an electromagnetic field,” Proc. Roy. Soc. Lond. A **173** (1939) 211-232.
- [120] K.T. Inoue, “Numerical Study of Length Spectra and Low-lying Eigenvalue Spectra of Compact Hyperbolic 3-manifolds,” Class. Quant. Grav. **18** (2001) 629-652, arXiv:math-ph/0011012.

- [121] C.J. Leininger, D.B. McReynolds, W.D. Neumann, and A.W. Reid, “Length and eigenvalue equivalence,” International Mathematics Research Notices 2007 (2007), rnm135-24, arXiv:math/0606343 [math.GT].
- [122] P. W. Anderson, “Absence of diffusion in certain random lattices,” Phys. Rev. **109** (1958) 1492-1505.
- [123] R. Camporesi and A. Higuchi, “Spectral functions and zeta functions in hyperbolic spaces,” J. Math. Phys. **35** (1994) 4217-4246.
- [124] R. Camporesi and A. Higuchi, “On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces,” J. Geom. Phys. **20** (1996) 1-18, arXiv:gr-qc/9505009.
- [125] S. Coleman, “Notes from Sidney Coleman’s Physics 253a,” arXiv:1110.5013 [physics.ed-ph].
- [126] A.D. Martin, W.J. Stirling, R.S. Thorne, G. Watt, “Parton distributions for the LHC,” Eur. Phys. J. **C63** (2009) 189-285, arXiv:0901.0002 [hep-ph].
http://projects.hepforge.org/mstwpdf/plots/mstw2008lo68cl_allpdfs.eps
- [127] The ATLAS Collaboration, “Search for new physics in multi-body final states at high invariant masses with ATLAS,” ATLAS-CONF-2010-088, 21 August 2010.
<http://cdsweb.cern.ch/record/1299103/files/ATLAS-CONF-2010-088.pdf>
- [128] The ATLAS Collaboration, “Search for high mass dilepton resonances in pp collisions at $\sqrt{s} = 7$ TeV with the ATLAS experiment,” ATLAS-CONF-2011-083, 6 June 2011.
<http://cdsweb.cern.ch/record/1356190/files/ATLAS-CONF-2011-083.pdf>
- [129] G. Aad *et al.* [ATLAS Collaboration], “Search for dilepton resonances in pp collisions at $\sqrt{s} = 7$ TeV with the ATLAS detector,” Phys. Rev. Lett. **107** (2011) 272002, arXiv:1108.1582 [hep-ex].
- [130] The CMS Collaboration, “Search for Heavy Resonances Decaying to Long-Lived Neutral Particles in the Displaced Lepton Channel,” CMS-PAS-EXO-11-004, 7 September 2011.
<http://cdsweb.cern.ch/record/1380311/files/EXO-11-004-pas.pdf>

- [131] G. Aad *et al.* [ATLAS Collaboration], “Search for New Physics in the Dijet Mass Distribution using 1 fb^{-1} of pp Collision Data at $\sqrt{s} = 7 \text{ TeV}$ collected by the ATLAS Detector,” Phys. Lett. B **708** (2012) 37-54, arXiv:1108.6311 [hep-ex].
- [132] S. Chatrchyan *et al.* [CMS Collaboration], “Search for signatures of extra dimensions in the diphoton mass spectrum at the Large Hadron Collider,” arXiv:1112.0688 [hep-ex].
- [133] http://en.wikipedia.org/wiki/Stokes'_theorem
- [134] D. Hundertmark, “A short introduction to Anderson localization,” in *Proceedings of the LMS Meeting on Analysis and Stochastics of Growth Processes and Interface Models, Bath, September 2006*, Oxford Univ. Press, Oxford, 2008, 194-218.
<http://www.math.uiuc.edu/~dirk/preprints/localization3.pdf>
- [135] A. Lagendijk, B. van Tiggelen, and D. S. Wiersma, “Fifty years of Anderson localization,” Physics Today **62(8)** (2009) 24-29.
http://physicstoday.org/resource/1/phtoad/v62/i8/p24_s1
- [136] S. S. Kondov, W. R. McGehee, J. J. Zirbel, and B. DeMarco, “Three-Dimensional Anderson Localization of Ultracold Matter,” Science **334** (2011) 66,
arXiv:1105.5368 [cond-mat.quant-gas].
- [137] A. F. Ioffe and A. R. Regel, “Non-Crystalline, Amorphous, and Liquid Electronic Semiconductors,” in *Progress in Semiconductors, Vol. 4*, edited by A. F. Gibson, F. A. Kroger, and R. E. Burgess, Wiley, New York, 1960, pp. 237-291.
- [138] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer Verlag, New York, 1990.
- [139] A. Selberg, “Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series,” J. Indian Math. Soc. (N.S.) **20** (1956) 47-87.
- [140] P. Lefèvre, T. S. Pettersson, *et al.* [The LHC Study Group], “The Large Hadron Collider: Conceptual Design,” CERN/AC/95-05(LHC), 20 October 1995.
<http://cdsweb.cern.ch/record/291782/files/cm-p00047618.pdf>

- [141] D. Acosta, G. L. Bayatian, *et al.* [CMS Collaboration], “CMS Physics, Technical Design Report, Volume I: Detector Performance and Software,” CERN/LHCC 2006-001, 2 February 2006.
<http://cdsweb.cern.ch/record/922757/files/lhcc-2006-001.pdf>
- [142] Chamonix 2012, Sessions 3 & 4, Strategy for 2012,
[http://indico.cern.ch/getFile.py/access?contribId=2&resId=1
&materialId=slides&confId=170230](http://indico.cern.ch/getFile.py/access?contribId=2&resId=1&materialId=slides&confId=170230)
- [143] The ATLAS Collaboration, “Atlas Detector and Physics Performance: Technical Design Report, Volume I,” ATLAS TDR 14, CERN/LHCC 99-14, 25 May 1999.
<http://cdsweb.cern.ch/record/391176/files/ce-0317330.pdf>
- [144] The ATLAS Collaboration, “Upper Limits on the Charge in Satellite Bunches for the October 2010 LHC Luminosity Calibration,” ATLAS-CONF-2011-049, 30 March 2011.
<http://cdsweb.cern.ch/record/1340989/files/ATLAS-CONF-2011-049.pdf>
- [145] The ATLAS Collaboration, “Luminosity Determination in pp Collisions at $\sqrt{s} = 7$ TeV using the ATLAS Detector in 2011,” ATLAS-CONF-2011-116, 19 August 2011.
<http://cdsweb.cern.ch/record/1376384/files/ATLAS-CONF-2011-116.pdf>
- [146] The ATLAS Collaboration, “Non-collision backgrounds as measured by the ATLAS detector during the 2010 proton-proton run,” ATLAS-CONF-2011-137, 20 September 2011.
<http://cdsweb.cern.ch/record/1383840/files/ATLAS-CONF-2011-137.pdf>
- [147] D. Olivito, for the ATLAS collaboration, “Searches for high mass dilepton resonances in pp collisions at $\sqrt{s} = 7$ TeV with the ATLAS Experiment,” arXiv:1109.0934 [hep-ex].
- [148] The ATLAS Collaboration, “Measurement of inclusive jet and dijet cross sections in proton-proton collision data at 7 TeV centre-of-mass energy using the ATLAS detector,” ATLAS-CONF-2011-047, 12 Jun 2011.
<http://cdsweb.cern.ch/record/1338578/files/ATLAS-CONF-2011-047.pdf>

- [149] The ATLAS Collaboration, “Search for high-mass states with one muon plus missing transverse momentum in proton-proton collisions at $\sqrt{s} = 7$ TeV with the ATLAS detector,” ATLAS-CONF-2011-082, 2 June 2011.
<http://cdsweb.cern.ch/record/1356189/files/ATLAS-CONF-2011-082.pdf>
- [150] G. Aad *et al.* [ATLAS Collaboration], “Search for New Physics in Dijet Mass and Angular Distributions in pp Collisions at $\sqrt{s} = 7$ TeV Measured with the ATLAS Detector,” New J. Phys. **13** (2011) 053044, arXiv:1103.3864 [hep-ex].
- [151] The ATLAS Collaboration, “Update of the Search for New Physics in the Dijet Mass Distribution in 163 pb⁻¹ of pp Collisions at $\sqrt{s} = 7$ TeV Measured with the ATLAS Detector,” ATLAS-CONF-2011-081, 2 June 2011.
<http://cdsweb.cern.ch/record/1355704/files/ATLAS-CONF-2011-081.pdf>
- [152] The ATLAS Collaboration, “Search for New Physics in Dijet Mass Distributions in 0.81 fb⁻¹ of pp Collisions at $\sqrt{s} = 7$ TeV,” ATLAS-CONF-2011-095, 21 July 2011.
<http://cdsweb.cern.ch/record/1369186/files/ATLAS-CONF-2011-095>
- [153] The ATLAS Collaboration, “Measurement of multi-jet cross-sections in proton-proton collisions at 7 TeV center-of-mass energy,” ATLAS-CONF-2011-043, 23 March 2011.
<http://cdsweb.cern.ch/record/1338572/files/ATLAS-CONF-2011-043.pdf>
- [154] The ATLAS Collaboration, “Characterization of Interaction-Point Beam Parameters Using the pp Event-Vertex Distribution Reconstructed in the ATLAS Detector at the LHC,” ATLAS-CONF-2010-027, 13 July 2010.
<http://cdsweb.cern.ch/record/1277659/files/ATLAS-CONF-2010-027.pdf>
- [155] S. Chatrchyan *et al.* [CMS Collaboration], “Performance of the CMS Hadron Calorimeter with Cosmic Ray Muons and LHC Beam Data,” JINST **5** (2010) T03012, arXiv:0911.4991 [physics.ins-det].
- [156] G. Aad *et al.* [The ATLAS Collaboration], “Studies of the performance of the ATLAS detector using cosmic-ray muons,” Eur. Phys. J. C **71** (2011) 1593, arXiv:1011.6665 [physics.ins-det].

- [157] The ATLAS Collaboration, “Performance of primary vertex reconstruction in proton-proton collisions at $\sqrt{s} = 7$ TeV in the ATLAS experiment,” ATLAS-CONF-2010-069, 28 July 2010.
<http://cdsweb.cern.ch/record/1281344/files/ATLAS-CONF-2010-069.pdf>
- [158] Maxima, a Computer Algebra System.
<http://maxima.sourceforge.net/>
- [159] K. Peeters, Cadabra: A field-theory motivated approach to computer algebra.
<http://cadabra.phi-sci.com/>
- [160] K. Peeters, “Introducing Cadabra: a symbolic computer algebra system for field theory problems,” arXiv:hep-th/0701238.
- [161] K. Peeters, “A field-theory motivated approach to symbolic computer algebra,” Comp. Phys. Commun. **176** (2007) 550-558, arXiv:cs/0608005 [cs.SC].
- [162] <http://www.gnuplot.info/>
- [163] G. Salam, Ordercite, a program to establish whether your bibliography is in the same order as the citations to it.
<http://www.lpthe.jussieu.fr/~salam/ordercite/>
- [164] J. van der Hoeven, GNU TeXmacs, a free “what you see is what you want” editing platform with special features for scientists.
<http://www.texmacs.org/>
- [165] <http://www.kde.org/>
- [166] <http://www.debian.org/>
- [167] Kile - an Integrated LaTeX Environment
<http://kile.sourceforge.net/>